A TREATISE ON

THE ANALYTIC GEOMETRY OF

THREE DIMENSIONS

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ANALYTIC GEOMETRY

OF

THREE DIMENSIONS

BY

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EDITOR'S PREFACE TO FIFTH EDITION.

VOLUME I.

IN order to avoid delay it has been thought advisable to publish this edition in two volumes. While preserving the substance of the fourth edition, I have added some new matter (generally enclosed in square brackets and in different type) giving brief accounts of methods or points of view which appear to me to be of interest and to fit in with the rest of the work.

The additions include illustrations of models of most of the different species of quadrics, with generators or lines of curvature (Chapter V.); articles or paragraphs on the analytical classification of real quadrics (88*a*), on projection and Fiedler's projective coordinates (144*c*), on the non-Euclidean theory of distance and angle (144*d*), and on the expression of twisted cubics and quartics by rational or elliptic parameters (333*a*, 347*a*, 348, 349).

In differential geometry my aim has been to form a closer connecting-link between Salmon's book and the more extensive and more purely analytical methods used by Bianchi, Darboux and others. I have therefore added articles on the now well-known Frenet-Serret formulæ, with some applications (368*a*), on

the intrinsic equations of a twisted curve (368*b*), on Bertrand curves (368*c*), and on the application of Gauss's parametric method to conformal representation, geodesic curvature and geodesic torsion (396*a*, 396*b*).

To the portion dealing with the differential geometry of curves on quadrics, I have added Staude's "thread-construction" for ellipsoids (421*a*), which is the three-dimensional analogue of Graves's theorem; and his definitions of confocal quadrics (421*b*) by means of "broken distances"—these are the analogues of the ordinary definitions of conics by means of focal radii. In the Golden Age of Euclidean geometry, analogues of these types were of great interest to men like Jacobi, MacCullagh, Chasles and M. Roberts, but Staude's constructions have virtually brought the subject to a conclusion. Staude's treatment is also an excellent illustration of the elementary and visible meaning of elliptic and hyper-elliptic integrals.

New matter is also contained in Arts. 80*a*, 80*b*, 88*a*, 159*a*, 172, 173, 176*a*, 261, 304, 384, and in various paragraphs throughout the book. Most of these additions are of the nature of commentaries. About 100 examples are added—many of them being solved—in order to illustrate the principles of the articles to which they are appended.

The numbering of the chapters and of the articles is the same as in the fourth edition, except in Chapter III., where the order has been somewhat changed, and in Arts. 172, 173.

The present edition has been published by the

direction of the Board of Trinity College, who appointed me as Editor in November, 1910.

REGINALD A. P. ROGERS.

TRINITY COLLEGE, DUBLIN,
November, 1911.

The Sixth Edition (1914) of Vol. I. is reprinted from the Fifth, with a few corrections, of which the most important are in Arts. 88*a*, 338 (Ex.), 344, 357.

Owing to the death of R. A. P. Rogers in October, 1923, the work of preparing this Seventh Edition of Vol. I. has been undertaken by me.

Beyond the correction of misprints, very little has been changed with the exception of a few paragraphs which were felt to be misleading.

The principal alterations will be found in Arts. 32 (Ex. 1), 40, 80, 88*a*, 114, 176*a*, 192, 218, 348, 368*a*, 396*a*.

CHARLES H. ROWE.

TRINITY COLLEGE, DUBLIN,
July, 1927.

PREFACE TO THE THIRD EDITION.

In the preface to the second edition of my *Higher Plane Curves*, I have explained the circumstances under which I obtained Professor Cayley's valuable help in the preparation of that volume. I have now very gratefully to acknowledge that the same assistance has been continued to me in the re-editing of the present work. The changes from the preceding edition are not so numerous here as in the case of the *Higher Plane Curves*, partly because the book not having been so long out of print required less alteration, partly because the size to which the volume had already swelled made it necessary to be sparing in the addition of new matter. Prof. Cayley having read all the proof sheets, the changes made at his suggestion are too numerous to be particularized; but the following are the parts which, on now looking through the pages, strike me as calling for special acknowledgment, as being entirely or in great measure derived from him; Arts.* 51-53 on the six coordinates of a line, the account of focal lines Art. 146, Arts.† 314-322 on Gauss's method of representing the coordinates of a point on a surface by two parameters. The discussion of Orthogonal Surfaces is taken from a manuscript memoir of Prof. Cayley's

* These articles have been altered in the present edition.
† Now Arts. 377-384.

Arts.* 332-337 nearly without alteration, and the following articles with some modifications of my own. Prof. Cayley has also contributed Arts. † 347 and 359 on Curves, Art.‡ 468 on Complexes, Arts. 567 to the end of the chapter on Quartics, and Arts.§ 600 to the end. Prof. Casey and Prof. Cayley had each supplied me with a short note on Cyclides, but I found the subject so interesting that I wished to give it fuller treatment. and had recourse to the original memoirs.

I have omitted the appendix on Quaternions which was given in the former editions, the work of Professors Kelland and Tait having now made information on this subject very easy to be obtained. I have also omitted the appendix on the order of Systems of Equations, which has been transferred to the Treatise on Higher Algebra.

I have, as on several former occasions, to acknowledge help given me, in reading the proof sheets, by my friends Dr. Hart, Mr. Cathcart and Dr. Fiedler.

Owing to the continued pressure of other engagements I have been able to take scarcely any part in the revision of this fourth edition. My friend, Mr. Cathcart, has laid me under the great obligation of taking the work almost entirely off my hands, and it is at his suggestion that some few changes have been made from the last edition.

TRINITY COLLEGE, DUBLIN,
Sept., 1882.

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* Arts. 476-479 in fourth edition.
† Arts. 316 and 328 in fourth edition.
‡ Art. 453 in fourth edition.
§ Art. 620 in fourth edition.

CONTENTS OF VOLUME I.

The following selected course is recommended to Junior Readers: The Theory of Surfaces of the Second Order, Arts. 1—144*b*, omitting articles specially indicated in footnotes. *Confocal Surfaces*, Arts. 157—170. *The Curvature of Quadrics*, Arts. 194—199. *The General Theory of Surfaces*, Chap. XI. *The Theory of Curves*, Arts. 314—323, 358—376.

CHAPTER I.

COORDINATES.

CHAPTER II.

INTERPRETATION OF EQUATIONS.

CHAPTER III.

THE PLANE AND RIGHT LINE.

CHAPTER IV.

PROPERTIES OF QUADRICS IN GENERAL.

CHAPTER V.

CLASSIFICATION OF QUADRICS.

CHAPTER VI.

PROPERTIES OF QUADRICS DEDUCED FROM SPECIAL FORMS OF THEIR EQUATIONS.

CHAPTER VII.

RECIPROCATION, DUALITY, ABRIDGED NOTATION AND PROJECTION.

CHAPTER VIII.

FOCI AND CONFOCAL SURFACES.

CHAPTER IX.

INVARIANTS AND COVARIANTS OF SYSTEMS OF QUADRICS.

CHAPTER X.

CONES AND SPHERO-CONICS.

CHAPTER XI.

GENERAL THEORY OF SURFACES (INTRODUCTORY CHAPTER). CURVATURE.

CHAPTER XII.

CURVES AND DEVELOPABLES.

ANALYTIC GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

COORDINATES.

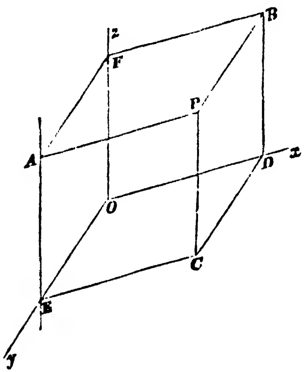
1. We have seen already how the position of a point C in a plane is determined, by referring it to two coordinate axes OX, OY drawn in the plane. To determine the position of any point P in space, we have only to add to our apparatus a third axis OZ not in the plane (see figure next page). Then, if we knew the distance measured parallel to the line OZ of the point P from the plane XOY, and also knew the x and y coordinates of the point C, where PC parallel to OZ meets the plane, it is obvious that the position of P would be completely determined.

Thus, if we were given the three equations $x = a$, $y = b$, $z = c$, the first two equations would determine the point C, and then drawing through that point a parallel to OZ, and taking on it a length $PC = c$, we should have the point P.

We have seen already how a change in the sign of a or b affects the position of the point C. In like manner the sign of c will determine on which side of the plane XOY the line PC is to be measured. If we conceive the plane XOY to be horizontal, it is customary to consider lines measured upwards as positive, and lines measured downwards as negative. In this case, then, the z of every point above that plane is counted as positive, and of every point below it as negative It is obvious that every point *on* the plane has its $z = 0$.

The angles between the axes may be any whatever; but the axes are said to be rectangular when the lines OX, OY are at right angles to each other, and the line OZ perpendicular to the plane XOY.

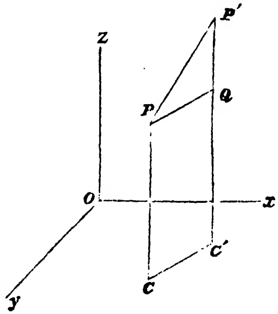
2. We have stated the method of representing a point in space, in the manner which seemed most simple for readers already acquainted with Plane Analytic Geometry. We proceed now to state the same more symmetrically. Our apparatus evidently consists of *three* coordinate axes OX, OY, OZ meeting in a point O, which, as in Plane Geometry, is called the origin. The three axes are called the axes of x, y, z respectively. These three *axes* determine also three coordinate *planes*, namely, the planes YOZ, ZOX, XOY, which we shall call the planes yz, zx, xy, respectively. Now since it is plain that $PA = CE = a$, $PB = CD = b$, we may say that the position of any point P is known if we are given its three coordinates; viz. PA drawn parallel to the axis of x to meet the plane yz, PB parallel to the axis of y to meet the plane zx, and PC parallel to the axis of z to meet the plane xy.



Again, since $OD = a$, $OE = b$, $OF = c$, the point given by the equations $x = a$, $y = b$, $z = c$, may be found by the following symmetrical construction: measure on the axis of x, the length $OD = a$, and through D draw the plane $PBCD$ parallel to the plane yz: measure on the axis of y, $OE = b$, and through E draw the plane $PACE$ parallel to zx: measure on the axis of z, $OF = c$, and through F draw the plane $PABF$ parallel to xy: the intersection of the three planes so drawn is the point P, whose construction is required.

3. The points A, B, C are called the *projections* of the point P on the three coordinate planes; and when the axes are rectangular they are its *orthogonal* projections. In what follows we shall be almost exclusively concerned with orthogonal projections, and therefore when we speak simply of projections, are to be understood to mean orthogonal projections, unless the contrary is stated. There are some properties of orthogonal projections which we shall often have occasion to employ, and which we therefore collect here, though we have given the proof of some of them already. (See *Conics*, Art. 368.)

The length of the orthogonal projection of a finite right line on any plane is equal to the line multiplied by the cosine of the angle which it makes with the plane.



The angle a line makes with a plane is measured by the angle which the line makes with its orthogonal projection on that plane. Let PC, $P'C'$ be drawn perpendicular to the plane XOY; and CC' is the orthogonal projection of the line PP' on that plane. Complete the rectangle by drawing PQ parallel to CC', and PQ will also be equal to CC'. But $PQ = PP' \cos P'PQ$.

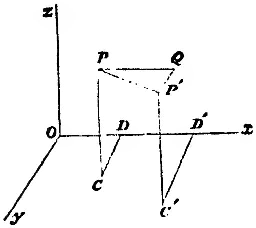
4. *The projection, on any plane, of any area in another plane is equal to the original area multiplied by the cosine of the angle between the planes.*

The angle between two planes is measured by the angle between the perpendiculars drawn in each plane to their line of intersection at any point of it. (It may also be measured by the angle between the perpendiculars let fall on the planes from any point.)

If ordinates of both figures be drawn perpendicular to the

intersection of the two planes, then, by the last article, every ordinate of the projection is equal to the corresponding ordinate of the original figure multiplied by the cosine of the angle between the planes. But it was proved (*Conics*, Art. 394), that when two figures are such that the ordinates corresponding to equal abscissæ have to each other a constant ratio, then the areas of the figures have to each other the same ratio.

5. The projection of a point on any *line* is the point where the line is met by a plane drawn through the point perpendicular to the line. Thus, in figure, p. 2, if the axes be rectangular, D, E, F are the projections of the point P on the three axes.



The projection of a finite right line upon another right line is equal to the first line multiplied by the cosine of the angle between the lines.

The angle between two lines which do not intersect, is measured by the angle between parallels to both drawn through any point. Let PP' be the given line, and DD' its projection on OX. Through P draw PQ parallel to OX to meet the plane $P'C'D'$; then since PQ is perpendicular to this plane, the angle PQP' is right, and $PQ = PP' \cos P'PQ$. But PQ and DD' are equal, since they are the intercepts made by two parallel planes on two parallel right lines.

When we speak of the angle between two lines, it is desirable to express without ambiguity whether we mean the acute or the obtuse angle which they make with each other. When therefore we speak of the angle between two lines (for instance PP' and CC' in the figure on p. 3), we shall understand that these lines are measured in the *directions* from P to P' and from C to C', and that PQ parallel to CC' is measured in the same direction. The angle then between the lines is $P'PQ$. But if we spoke of the angle between PP' and $C'C$, we should draw the parallel PQ' in the opposite direction to PQ, and the angle expressed would be $P'PQ'$, the supplement of $P'PQ$.

When we speak of the angles made by any line OP with the axes, we

shall always mean the angles between OP and the *positive* directions of the axes, viz. OX, OY, OZ.

6. *If there be any three points P, P', P'', the projection of PP'' on any line will be equal to the sum of the projections on that line of PP' and $P'P''$.*

Let the projections of the three points be D, D', D'', then if D' lie between D and D'', DD'' is evidently the sum of DD' and $D'D''$. If D'' lie between D and D', DD'' is the difference of DD' and $D'D''$; but since the direction from D' to D'' is the opposite of that from D to D', DD'' is still the algebraic sum of DD' and $D'D''$. It may be otherwise seen that the projection of $P'P''$ is in the latter case to be taken with a negative sign, from the consideration that in this case the length of the projection is found by multiplying $P'P''$ by the cosine of an *obtuse* angle (see Art. 5).

In general, if there be any number of points P, P', P'', P''', &c., the projection of PP''' on any line is equal to the sum of the projections of PP', $P'P''$, $P''P'''$, &c. The theorem may also be expressed in the form that the sum of the projections on any line of the sides of a closed polygon $= 0$.

7. We shall frequently have occasion to make use of the following particular case of the preceding:

If the coordinates of any point P be projected on any line, the sum of the three projections is equal to the projection of the radius vector on that line.

For consider the points O, D, C, P (see figure, p. 2) and the projection of OP must be equal to the sum of the projections of OD $(=x)$, DC $(=y)$, and CP $(=z)$.

8. Having established those principles concerning projections which we shall constantly have occasion to employ, we return now to the more immediate subject of this chapter.

The coordinates of the point P dividing the distance between two points P' $(x'y'z')$, P'' $(x''y''z'')$ so that $P'P : PP'' :: m : l$, are

$$x = \frac{lx' + mx''}{l+m}, \quad y = \frac{ly' + my''}{l+m}, \quad z = \frac{lz' + mz''}{l+m}.$$

The proof is precisely the same as that given at *Conics*, Art. 7, for the corresponding theorem in Plane Analytic Geometry. The lines PM, QN in the figure there given now represent the ordinates drawn from the two points to any one of the coordinate planes.

If we consider the ratio $l : m$ as indeterminate, we have the coordinates of *any* point on the line joining the two given points.

9. *Any side of a triangle $P''' P''$ is cut in the ratio $m : n$, and the line joining this point to the opposite vertex P' is cut in the ratio $m + n : l$, to find the coordinates of the point of section.*

Ans.

$$x = \frac{lx' + mx'' + nx'''}{l + m + n}, \quad y = \frac{ly' + my'' + ny'''}{l + m + n}, \quad z = \frac{lz' + mz'' + nz'''}{l + m + n}.$$

This is proved as in Plane Analytic Geometry (see *Conics*, Art. 7).

If we consider l, m, n as indeterminate, we have the coordinates of *any* point in the plane determined by the three points.

Ex. The lines joining middle points of opposite edges of a tetrahedron meet in a point. The x's of two such middle points are $\frac{1}{2}(x' + x'')$, $\frac{1}{2}(x''' + x'''')$, and the x of the middle point of the line joining them is $\frac{1}{4}(x' + x'' + x''' + x'''')$. The other coordinates are found in like manner, and their symmetry shows that this is also a point on the line joining the other middle points. Through this same point will pass the line joining each vertex to the centre of gravity of the opposite triangle. For the x of one of these centres of gravity is $\frac{1}{3}(x' + x'' + x''')$, and if the line joining this to the opposite vertex be cut in the ratio of $3 : 1$, we get the same value as before.

10. *To find the distance between two points P, P', whose rectangular coordinates are $x'y'z'$, $x''y''z''$.*

Evidently (see figure, p. 3) $PP'^2 = PQ^2 + QP'^2$. But $QP' = z' - z''$, and $PQ^2 = CC'^2$ is by Plane Analytic Geometry $= (x' - x'')^2 + (y' - y'')^2$. Hence

$$PP'^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2.$$

Cor. The distance of any point $x'y'z'$ from the origin is given by the equation

$$OP^2 = x'^2 + y'^2 + z'^2.$$

11. The position of a point is sometimes expressed by its radius vector and the angles it makes with three rectangular axes. Let these angles be α, β, γ. Then since the coordinates x, y, z are the projections of the radius vector on the three axes, we have

$$x = \rho \cos \alpha,\ y = \rho \cos \beta,\ z = \rho \cos \gamma.$$

And, since $x^2 + y^2 + z^2 = \rho^2$, the three cosines (which are sometimes called the direction-cosines of the radius vector) are connected by the relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.*$$

Moreover (compare Art. 7), $x \cos \alpha + y \cos \beta + z \cos \gamma = \rho$.

The position of a point P (see fig. p. 4) is also sometimes expressed by the following *polar coordinates*—the radius vector, the angle γ which the radius vector makes with a fixed axis OZ, and the angle COD $(= \phi)$ which OC the projection of the radius vector on a plane perpendicular to OZ makes with a fixed line OX in that plane. Since then $OC = \rho \sin \gamma$, the formulæ for transforming from rectangular to these polar coordinates are

$$x = \rho \sin \gamma \cos \phi,\ y = \rho \sin \gamma \sin \phi,\ z = \rho \cos \gamma.$$

[The angles γ and ϕ may be represented as spherical arcs. Draw a sphere of unit radius with the origin as centre meeting the axes in $X'Y'Z'$, and the radius vector OP in P'. Let the great-circle $Z'P'$ meet the great-circle $X'Y'$ in M'. Then the arc $M'X' = \phi$, and the arc $P'Z' = \gamma$. ϕ is often termed the

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* I have followed the usual practice in denoting the position of a line by these angles, but in one point of view there would be an advantage in using instead the complementary angles, namely, the angles which the line makes with the coordinate planes. This appears from the corresponding formulæ for oblique axes, which I have not thought it worth while to give in the text, as we shall not have occasion to use them afterwards. Let α, β, γ be the angles which a line makes with the planes yz, zx, xy, and let A, B, C be the angles which the axis of x makes with the plane of yz, of y with the plane of zx, and of z with the plane of xy, then the formulæ which correspond to those in the text are

$$x \sin A = \rho \sin \alpha,\ y \sin B = \rho \sin \beta,\ z \sin C = \rho \sin \gamma.$$

These formulæ are proved by the principle of Art. 7. If we project on a line perpendicular to the plane of yz, since the projections of y and of z on this line vanish, the projection of x must be equal to that of the radius vector, and the angles made by x and ρ with this line are the complements of A and α.

longitude and γ the co-latitude of the point P, the plane OXY or the corresponding great-circle $X'Y'$ being termed the equator.

The position of a point may also be determined by *cylindrical coordinates*. These are r, z, and ϕ, where r is the length of the perpendicular from the point on the axis of z.

These four systems of coordinates, viz. Cartesian (x, y, z), radio-angular (ρ, α, β, γ), polar (ρ, γ, ϕ), and cylindrical (r, z, ϕ), have the property that in real space only one point has a given set of coordinates, and conversely, if a point is given, its coordinates are uniquely determined. But if we took as coordinates for example, ρ, x, y, or r_1, r_2, r_3 (the distances of P from three fixed points), then in each case *two* points would correspond to given values of the coordinates.]

12. *The square of the area of any plane figure is equal to the sum of the squares of its projections on three rectangular planes.*

Let the area be A, and let a perpendicular to its plane make angles α, β, γ with the three axes; then (Art. 4) the projections of this area on the planes yz, zx, xy respectively, are $A\cos\alpha$, $A\cos\beta$, $A\cos\gamma$. And the sum of the squares of these three $= A^2$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

13. *To express the cosine of the angle θ between two lines OP, OP' in terms of the direction-cosines of these lines.*

We have proved (Art. 10) that

$$PP'^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

But also $$PP'^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\theta.$$

And since $$\rho^2 = x^2 + y^2 + z^2,\quad \rho'^2 = x'^2 + y'^2 + z'^2,$$

we have $$\rho\rho' \cos\theta = xx' + yy' + zz',$$

or $$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Cor. The condition that two lines should be at right angles to each other is

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$$

14. The following formula is also sometimes useful:

$$\sin^2\theta = (\cos\beta\cos\gamma' - \cos\gamma\cos\beta')^2 + (\cos\gamma\cos\alpha' - \cos\alpha\cos\gamma')^2 + (\cos\alpha\cos\beta' - \cos\beta\cos\alpha')^2.$$

This may be derived from the following elementary theorem for the sum of the squares of three determinants (*Lessons on Higher Algebra*, Art. 26), which can be verified at once by actual expansion,

$$(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2$$
$$= (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2) - (aa' + bb' + cc')^2.$$

For when a, b, c; a', b', c' are the direction-cosines of two lines, the right-hand side becomes $1 - \cos^2\theta$.

Ex. To find the perpendicular distance from a point $x'y'z'$ to a line through the origin whose direction-angles are α, β, γ.

Let P be the point $x'y'z'$, OQ the given line, PQ the perpendicular, then it is plain that $PQ = OP \sin POQ$; and using the value just obtained for $\sin POQ$, and remembering that $x' = OP \cos \alpha'$, etc., we have

$$PQ^2 = (y' \cos\gamma - z' \cos\beta)^2 + (z' \cos\alpha - x' \cos\gamma)^2 + (x' \cos\beta - y' \cos\alpha)^2.$$

15. *To find the direction-cosines of a line perpendicular to two given lines, and therefore perpendicular to their plane.*

Let $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ be the direction-angles of the given lines, and $\alpha\beta\gamma$ of the required line, then we have to find $\alpha\beta\gamma$ from the three equations

$$\cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' = 0,$$
$$\cos\alpha \cos\alpha'' + \cos\beta \cos\beta'' + \cos\gamma \cos\gamma'' = 0,$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

From the first two equations we can easily derive, by eliminating in turn $\cos\alpha$, $\cos\beta$, $\cos\gamma$,

$$\lambda \cos\alpha = \cos\beta' \cos\gamma'' - \cos\beta'' \cos\gamma',$$
$$\lambda \cos\beta = \cos\gamma' \cos\alpha'' - \cos\gamma'' \cos\alpha',$$
$$\lambda \cos\gamma = \cos\alpha' \cos\beta'' - \cos\alpha'' \cos\beta',$$

where λ is indeterminate; and substituting in the third equation, we get (see Art. 14), if θ be the angle between the two given lines,

$$\lambda^2 = \sin^2\theta.$$

This result may be also obtained as follows: take any two points P, Q, or $x'y'z'$, $x''y''z''$, one on each of the two given lines. Now double the area of the projection on the plane of xy of the triangle POQ, is (see *Conics*, Art. 36) $x'y'' - y'x''$, or $\rho'\rho''(\cos\alpha' \cos\beta'' - \cos\alpha'' \cos\beta')$. But double the area of the triangle is $\rho'\rho'' \sin\theta$, and therefore the projection on the plane of xy is $\rho'\rho'' \sin\theta \cos\gamma$. Hence, as before,

$$\sin\theta \cos\gamma = \cos\alpha' \cos\beta'' - \cos\alpha'' \cos\beta',$$

and in like manner

$$\sin\theta \cos\alpha = \cos\beta' \cos\gamma'' - \cos\beta'' \cos\gamma';$$
$$\sin\theta \cos\beta = \cos\gamma' \cos\alpha'' - \cos\gamma'' \cos\alpha'.$$

Transformation of Coordinates.

16. *To transform to parallel axes through a new origin, whose coordinates referred to the old axes, are x', y', z'.*

The formulæ of transformation are (as in Plane Geometry)

$$x = X + x',\ y = Y + y',\ z = Z + z'.$$

For let a line drawn through the point P parallel to one of the axes (for instance z) meet the old plane of xy in a point C, and the new in a point C'; then $PC = PC' + C'C$.

But PC is the old z, PC' is the new z; and since parallel planes make equal intercepts on parallel right lines, $C'C$ must be equal to the line drawn through the new origin O parallel to the axis of z, to meet the old plane of xy.

17. *To pass from a rectangular system of axes to another system of axes having the same origin.*

Let the angles made by the new axes of x, y, z with the old axes be a, β, γ; a', β', γ'; a'', β'', γ'' respectively. Then if we project the new coordinates on one of the old axes, the sum of the three projections will (Art. 7) be equal to the projection of the radius vector, which is the corresponding old coordinate. Thus we get the three equations

$$\left.\begin{aligned} x &= X\cos a + Y\cos a' + Z\cos a'' \\ y &= X\cos\beta + Y\cos\beta' + Z\cos\beta'' \\ z &= X\cos\gamma + Y\cos\gamma' + Z\cos\gamma'' \end{aligned}\right\} \ldots\ldots\ldots (A).$$

We have, of course, (Art. 11)

$$\cos^2 a + \cos^2\beta + \cos^2\gamma = 1,\ \cos^2 a' + \cos^2\beta' + \cos^2\gamma' = 1,$$
$$\cos^2 a'' + \cos^2\beta'' + \cos^2\gamma'' = 1 \ldots\ldots\ldots\ldots (B).$$

Let λ, μ, ν be the angles between the new axes of y and z, of z and x, of x and y respectively, then (Art. 13)

$$\left.\begin{aligned} \cos\lambda &= \cos a' \cos a'' + \cos\beta' \cos\beta'' + \cos\gamma' \cos\gamma'' \\ \cos\mu &= \cos a'' \cos a + \cos\beta'' \cos\beta + \cos\gamma'' \cos\gamma \\ \cos\nu &= \cos a \cos a' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' \end{aligned}\right\} \ldots (C).$$

18. If the new axes be also rectangular, we have therefore

$$\left.\begin{aligned} \cos a' \cos a'' + \cos\beta' \cos\beta'' + \cos\gamma' \cos\gamma'' &= 0 \\ \cos a'' \cos a + \cos\beta'' \cos\beta + \cos\gamma'' \cos\gamma &= 0 \\ \cos a \cos a' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' &= 0 \end{aligned}\right\} \ldots (D).$$

When the new axes are rectangular, since a, a', a'' are the angles made by the old axis of x with the new axes, &c. we must have

$$\cos^2 a+\cos^2 a'+\cos^2 a''=1,\ \cos^2\beta+\cos^2\beta'+\cos^2\beta''=1.$$
$$\cos^2\gamma+\cos^2\gamma'+\cos^2\gamma''=1\ \ldots\ldots\ldots\ldots\ (E),$$

$$\left.\begin{aligned}\cos\beta\cos\gamma+\cos\beta'\cos\gamma'+\cos\beta''\cos\gamma''=0\\ \cos\gamma\cos a+\cos\gamma'\cos a'+\cos\gamma''\cos a''=0\\ \cos a\cos\beta+\cos a'\cos\beta'+\cos a''\cos\beta''=0\end{aligned}\right\}\ldots(F),$$

and the new coordinates expressed in terms of the old are

$$\left.\begin{aligned}X&=x\cos a\ +y\cos\beta\ +z\cos\gamma\\ Y&=x\cos a'\ +y\cos\beta'\ +z\cos\gamma'\\ Z&=x\cos a''+y\cos\beta''+z\cos\gamma''\end{aligned}\right\}\ldots\ldots(G).$$

The two corresponding systems of equations A and G may be briefly expressed by the diagram

	X	Y	Z
x	a	a'	a''
y	β	β'	β''
z	γ	γ'	γ''

[Ex. Prove algebraically that the set of equations E, F are equivalent to the set of equations B, D.]

19. If we square and add equations (A) (Art. 17), attending to equations (C), we find

$$x^2+y^2+z^2=X^2+Y^2+Z^2+2YZ\cos\lambda+2ZX\cos\mu+2XY\cos\nu.$$

Thus we obtain the radius vector from the origin to any point expressed in terms of the oblique coordinates of that point. It is proved in like manner that the square of the distance between two points, the axes being oblique, is

$$(x'-x'')^2+(y'-y'')^2+(z'-z'')^2+2(y'-y'')(z'-z'')\cos\lambda$$
$$+2(z'-z'')(x'-x'')\cos\mu+2(x'-x'')(y'-y'')\cos\nu.$$

* As we rarely require in practice the formulæ for transforming from one set of oblique axes to another, we only give them in a note.

Let A, B, C have the same meaning as at note, p. 7, and let a, β, γ; a', β', γ'; a'', β'', γ'' be the angles made by the new axes with the old coordin-

20. *The degree of any equation between the coordinates is not altered by transformation of coordinates.*

This is proved, as at *Conics*, Art. 11, from the consideration that the expressions given (Arts. 16, 17) for x, y, z, only involve the new coordinates *in the first degree.*

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ate *planes*; then by projecting on lines perpendicular to the old coordinate planes, as in the note referred to, we find

$$x \sin A = X \sin \alpha + Y \sin \alpha' + Z \sin \alpha'',$$
$$y \sin B = X \sin \beta + Y \sin \beta' + Z \sin \beta'',$$
$$z \sin C = X \sin \gamma + Y \sin \gamma' + Z \sin \gamma''.$$

CHAPTER II.

INTERPRETATION OF EQUATIONS.

21. It appears from the construction of Art. 1 that if we were given merely the two equations $x = a$, $y = b$, and if the z were left indeterminate, the two given equations would determine the point C, and we should know that the point P lay *somewhere* on the line PC. These two equations then are considered as representing that right line, it being the locus of all points whose $x = a$, and whose $y = b$. We learn then that any two equations of the form $x = a$, $y = b$ represent a right line parallel to the axis of z. In particular, the equations $x = 0$, $y = 0$ represent the axis of z itself. Similarly for the other axes.

Again, if we were given the single equation $x = a$, we could determine nothing but the point D. Proceeding, as at the end of Art. 2, we should learn that the point P lay *somewhere* in the plane $PBCD$, but its position in that plane would be indeterminate. This plane then being the locus of all points whose $x = a$, is represented analytically by that equation. We learn then that any equation of the form $x = a$ represents a plane parallel to the plane yz. In particular, the equation $x = 0$ denotes the plane yz itself. Similarly, for the other two coordinate planes.

22. In general, *any single equation between the coordinates represents a surface of some kind; any two simultaneous equations between them represent a line of some kind, either straight or curved; and any three equations denote one or more points.*

I. If we are given a *single* equation, we may take for x and y any arbitrary values; and then the given equation solved for z will determine one or more corresponding values of z. In other words, if we take arbitrarily any point C in the plane of xy, we can always find on the line PC one or more points whose coordinates will satisfy the given equation. The assemblage then of points so found on the lines PC will form a surface which will be the geometrical representation of the given equation (see *Conics*, Art. 16).

II. When we are given *two* equations, we can, by eliminating z and y alternately between them, throw them into the form $y = \phi(x)$, $z = \psi(x)$. If then we take for x any arbitrary value, the given equations will determine corresponding values for y and z. In other words, we can no longer take the point C *anywhere* on the plane of xy, but this point is limited to a certain locus represented by the equation $y = \phi(x)$. Taking the point C anywhere on this locus, we determine as before on the line PC a number of points P, the assemblage of which is the locus represented by the two equations. And since the points C, which are the projections of these latter points, lie on a certain line, straight or curved, it is plain that the points P must also lie on a line of some kind, though of course they do not necessarily lie all in any one plane.

Otherwise thus: when two equations are given, we have seen in the first part of this article that the locus of points whose coordinates satisfy either equation separately is a surface. Consequently, the locus of points whose coordinates satisfy *both* equations is the assemblage of points common to the two surfaces which are represented by the two equations considered separately: that is to say, the locus is the line of intersection of these surfaces.

III. When *three* equations are given, they are in general sufficient to determine absolutely the values of the three unknown quantities x, y, z, and therefore the given equations represent one or more *points*. Since each equation taken separately represents a surface, it follows hence that

any three surfaces have one or more common points of intersection, real or imaginary.

23. Surfaces, like plane curves, are classed according to the degrees of the equations which represent them. Since every point in the plane of xy has its $z=0$, if in any equation we make $z=0$, we get the relation between the x and y coordinates of the points in which the plane xy meets the surface represented by the equation: that is to say, we get the equation of the plane curve of section, and it is obvious that the equation of this curve will be in general of the same degree as the equation of the surface. It is evident, in fact, that the degree of the equation of the section cannot be *greater* than that of the surface, but it appears at first as if it might be *less*. For instance, the equation

$$zx^2 + ay^2 + b^2x = c^3$$

is of the third degree; but when we make $z=0$, we get an equation of the second degree. But since the original equation would have been unmeaning if it were not homogeneous, every term must be of the third dimension in some linear unit (see *Conics*, Art. 69), and therefore when we make $z=0$, the remaining terms must still be regarded as of three dimensions. They will form an equation of the second degree multiplied by a constant, and denote (see *Conics*, Art. 67) a conic and a line at infinity. If then we take into account lines at infinity, we may say that the section of a surface of the order n by the plane of xy will be *always* of the order n; and since any plane may be made the plane of xy, and since transformation of coordinates does not alter the degree of an equation, we learn that *every plane section of a surface of the order n is a curve of the order n*.*

In like manner it is proved that *every right line meets a surface of the order n in n points*. The right line may be made the axis of z, and the points where it meets the surface

* [The argument is based on the principle that a root of an equation of the n^{th} degree in a variable t becomes greater than any assignable magnitude as the coefficient of t^n is diminished towards zero.]

are found by making $x=0$, $y=0$ in the equation of the surface, when in general we get an equation of the degree n to determine z. If the degree of the equation happened to be less than n, it would only indicate that some of the n points where the line meets the surface are at infinity (*Conics*, Art. 135).

24. *Curves in space* are classified according to the number of points in which they are met by any plane. *Two equations of the degrees m and n respectively represent a curve of the order mn.* For the surfaces represented by the equations are cut by any planes in curves of the orders m and n respectively, and these curves intersect in mn points.

Conversely, if the degree of a curve be decomposed in any manner into the factors m, n, then the curve *may be* the intersection of two surfaces of the degrees m, n respectively; and it is in this case said to be a complete intersection. But not every curve is a complete intersection: in particular we have curves, the degree of which is a prime number, which are not plane curves.

Three equations of the degrees m, n, and p in general denote mnp points. This follows from the theory of elimination, since if we eliminate y and z between the equations, we obtain an equation of the degree mnp to determine x (see *Higher Algebra*, Arts. 73, 78). This proves also that *three surfaces of the orders m, n, p in general intersect in mnp points*, except when they all pass through the same curve.

25. If an equation only contain two of the variables, $\phi(x, y)=0$, the learner might at first suppose that it represents a curve in the plane of xy, and so that it forms an exception to the rule that it requires *two* equations to represent a curve. But it must be remembered that the equation $\phi(x, y)=0$ will be satisfied not only for any point of this curve in the plane of xy, but also for any other point having the same x and y though a different z; that is to say, for any point of the surface generated by a right line moving

along this curve, but remaining parallel to the axis of z.* The curve in the plane of xy can only be represented by *two* equations, namely, $z = 0$, $\phi(x, y) = 0$.

If an equation contain only *one* of the variables, x, we know by the theory of equations that it may be resolved into n factors of the form $x - a = 0$, and therefore (Art. 21) that it represents n planes parallel to one of the coordinate planes.†

* A surface generated by a right line moving parallel to itself is called a *cylindrical* surface.

† [The surfaces considered by Salmon are *algebraic*, i.e. they are represented (after rationalization if necessary) by equations containing a finite number of terms of the form $x^p y^q z^r$ where p, q, r are positive integers. But —as in the case of plane curves—equations involving *transcendental* functions of the coordinates (e.g. sines and exponentials) also represent surfaces. Many of the infinitesimal properties of transcendental surfaces can be studied by the same methods as those of algebraic surfaces by expanding the transcendental functions in converging series, but such surfaces have no determinate finite order; e.g. the surface $y + z = \sin x$ meets the axis of x in an infinite number of points.]

CHAPTER III.

THE PLANE AND THE RIGHT LINE.

The Plane.

26. In the discussion of equations we commence of course with equations of the first degree, and the first step is to prove that *every equation of the first degree represents a plane*, and conversely, that *the equation of a plane is always of the first degree*. We commence with the latter proposition, which may be established in two or three different ways.

In the first place we have seen (Art. 21) that the plane of xy is represented by an equation of the first degree, viz. $z=0$; and transformation to any other axes cannot alter the *degree* of this equation (Art. 20).

We might arrive at the same result by forming the equation of the plane determined by three given points, which we can do by eliminating l, m, n from the three equations given in Art. 9, when we should arrive at an equation of the first degree. The following method, however, of expressing the equation of a plane leads to one of the forms most useful in practice.

27. *To find the equation of a plane, the perpendicular on which from the origin* $=p$, *and makes angles* α, β, γ *with the axes.*

The length of the projection on the perpendicular of the radius vector to any point of the plane is of course $=p$, and (Art. 7) this is equal to the sum of the projections on that

line of the three coordinates. Hence we obtain for the equation of the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.^*$$

28. Now, conversely, any equation of the first degree

$$Ax + By + Cz + D = 0$$

can be reduced to the form just given, by dividing it by a factor R. We are to have $A = R \cos \alpha$, $B = R \cos \beta$, $C = R \cos \gamma$, whence, by Art. 11, R is determined to be $= \surd (A^2 + B^2 + C^2)$. Hence any equation $Ax + By + Cz + D = 0$ may be identified with the equation of a plane, the perpendicular on which from the origin $= \dfrac{-D}{\surd (A^2 + B^2 + C^2)}$, and makes angles with the axes whose cosines are A, B, C, respectively divided by the same square root. We may give to the square root the sign which will make the perpendicular positive, and then the signs of the cosines will determine whether the angles which the perpendicular makes with the positive directions of the axes are acute or obtuse.

[Ex. Prove by pure Geometry that the equation $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ is satisfied by any point on a plane making intercepts a, b, c on the axes (oblique or rectangular).]

29. *To find the angle between two planes.*

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0.$$

The angle between the planes is the same as the angle between the perpendiculars on them from the origin. By the last article we have the angles these perpendiculars make with the axes, and thence, Arts. 13, 14, we have

$$\cos \theta = \frac{AA' + BB' + CC'}{\{\surd(A^2 + B^2 + C^2)\ (A'^2 + B'^2 + C'^2)\}},$$

$$\sin^2 \theta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(A^2 + B^2 + C^2)\ (A'^2 + B'^2 + C'^2)}.$$

Hence the condition that the planes should cut at right angles is $AA' + BB' + CC' = 0$.

* In what follows we suppose the axes rectangular, but this equation is true whatever be the axes.

They will be parallel if we have the conditions

$$BC' = B'C,\ CA' = C'A,\ AB' = A'B;$$

in other words, if the coefficients A, B, C be proportional to A', B', C', in which case it is manifest from the last article that the directions of the perpendiculars on both will be the same.

30. *To express the equation of a plane in terms of the intercepts a, b, c, which it makes on the axes.*

The intercept made on the axis of x by the plane

$$Ax + By + Cz + D = 0$$

is found by making y and z both $= 0$, when we have $Aa + D = 0$. And similarly, $Bb + D = 0$, $Cc + D = 0$. Substituting in the general equation the values just found for A, B, C, it becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If in the general equation any term be wanting, for instance if $A = 0$, the point where the plane meets the axis of x is at infinity, or the plane is parallel to the axis of x. If we have both $A = 0$, $B = 0$, then the axes of x and y meet at infinity the given plane, which is therefore parallel to the plane of xy (see also Art. 21). If we have $A = 0$, $B = 0$, $C = 0$, all three axes meet the plane at infinity, and we see, as at *Conics*, Art. 67, that an equation $0\,.\,x + 0\,.\,y + 0\,.\,z + D = 0$ must be taken to represent a plane at infinity.*

31. *To find the equation of the plane determined by three points.*

Let the equation be $Ax + By + Cz + D = 0$; then, since this is to be satisfied by the coordinates of each of the given points, A, B, C, D must satisfy the equations

$$Ax' + By' + Cz' + D = 0,\ Ax'' + By'' + Cz'' + D = 0,$$
$$Ax''' + By''' + Cz''' + D = 0.$$

Eliminating A, B, C, D between the four equations, the result is the determinant

* [This means that if D differs from zero the distance of the plane from the origin may be made greater than any assignable length by diminishing A, B and C towards zero.]

$$\begin{vmatrix} x, & y, & z, & 1 \\ x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & z''', & 1 \end{vmatrix} = 0.$$

Expanding this by the common rule, the equation is

$$\begin{aligned} &x\,\{y'(z''-z''') + y''(z'''-z') + y'''(z'-z'')\} \\ &+ y\,\{z'(x''-x''') + z''(x'''-x') + z'''(x'-x'')\} \\ &+ z\,\{x'(y''-y''') + x''(y'''-y') + x'''(y'-y'')\} \\ &= x'(y''z'''-y'''z'') + x''(y'''z'-y'z''') + x'''(y'z''-y''z'). \end{aligned}$$

If we consider x, y, z as the coordinates of any fourth point, we have the *condition that four points should lie in one plane.*

32. The coefficients of x, y, z in the preceding equation are evidently double the areas of the projections on the coordinate planes of the triangle formed by the three points.

If now we take the equation (Art. 27)

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

and multiply it by twice A (A being the area of the triangle formed by the three points), the equation will become identical with that of the last article, since $A \cos \alpha$, $A \cos \beta$, $A \cos \gamma$, are the projections of the triangle on the coordinate planes (Art. 4). The absolute term then must be the same in both cases. Hence the quantity

$$x'(y''z'''-y'''z'') + x''(y'''z'-y'z''') + x'''(y'z''-y''z')$$

represents double the area of the triangle formed by the three points multiplied by the perpendicular on its plane from the origin; or, in other words, *six times the volume of the triangular pyramid, whose base is that triangle, and whose vertex is the origin.*

We can at once express A itself in terms of the coordinates of the three points by Art. 12, and must have $4A^2$ equal to the sum of the squares of the coefficients of x, y, and z, in the equation of the last article.

Ex. 1. To express the volume of a tetrahedron in terms of three concurrent edges and the angle between them.

If in the preceding values we substitute for x', y', z' the values $\rho' \cos \alpha'$,

$\rho' \cos\beta'$, $\rho' \cos\gamma'$, &c., we find that six times the volume of this pyramid $= \rho'\rho''\rho''$ multiplied by the determinant

$$\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma' \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'' \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''' \end{vmatrix}.$$

Now let us suppose the three radii vectores cut by a sphere whose radius is unity, having the origin for its centre, and meeting it in a spherical triangle $R'R''R'''$. Then if a denote the side $R'R''$, and p the perpendicular on it from R''', six times the volume of the pyramid will be $\rho'\rho''\rho'''$ sin a sin p; for $\rho'\rho''$ sin a is double the area of one face of the pyramid, and ρ''' sin p is the perpendicular on it from the opposite vertex. It follows then that the determinant above written is equal to double the function

$$\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}$$

of the sides of the above-mentioned spherical triangle. The same thing may be proved by forming the square of the same determinant according to the ordinary rule; when if we write

$$\cos\alpha'' \cos\alpha''' + \cos\beta'' \cos\beta''' + \cos\gamma'' \cos\gamma''' = \cos a, \text{ \&c.},$$

we get

$$\begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1 \end{vmatrix},$$

which expanded is $1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c$, which is known to have the value in question.

It is important to notice that, when the three lines are at right angles, the value of the determinant written at the beginning of this example is ± 1. In fact, as above, its square is unity. [The sign may be fixed as follows. By the positive direction of a line whose direction cosines are l, m, n we mean the direction from the origin to the point (l, m, n). Suppose now that the positive directions of the three lines, taken in the order in which they figure in the determinant, have the same relative orientation as the positive directions of the axes of x, y, and z respectively, so that it is possible to move the three straight lines as a rigid body and bring their positive directions into coincidence with the positive directions of the coordinate axes in the order mentioned. In this case the determinant is $+1$, for, if the movement be effected continuously, the determinant must constantly retain one of the two possible values ± 1, so that its original value must be the same as its final value which is $+1$.]

It is easily seen that when the determinant has the positive value, each of its elements is equal to the corresponding first minor with the proper sign attached.

Ex. 2. If the coordinates are oblique six times the volume of the pyramid is equal to

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}$$

multiplied by the sine of the angle between the axes of y and z, multiplied by the sine of the angle between the axis of x and the plane of $y\,z$.]

33. *To find the length of the perpendicular from a given point $x'y'z'$ on a given plane,* $x \cos \alpha + y \cos \beta + z \cos \gamma = p$.

If we draw through $x'y'z'$ a plane parallel to the given plane, and let fall on the two planes a common perpendicular from the origin, then the intercept on this line will be equal to the length of the perpendicular required, since parallel planes make equal intercepts on parallel lines. But the length of the perpendicular on the plane through $x'y'z'$ is, by definition (Art. 5), the projection on that perpendicular of the radius vector to $x'y'z'$, and therefore (Art. 27) is equal to

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma.$$

The length required is therefore

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p.$$

N.B.—This supposes the perpendicular on the plane through $x'y'z'$ to be greater than p; or, in other words, that $x'y'z'$ and the origin are on opposite sides of the plane. If they were on the same side, the length of the perpendicular would be $p - (x' \cos \alpha + y' \cos \beta + z' \cos \gamma)$. If the equation of the plane had been given in the form $Ax + By + Cz + D = 0$, it is reduced, as in Art. 28, to the form here considered, and the length of the perpendicular is found to be

$$\frac{Ax' + By' + Cz' + D}{\sqrt{(A^2 + B^2 + C^2)}}.$$

It is plain that all points for which $Ax' + By' + Cz' + D$ has the same sign as D, will be on the same side of the plane as the origin; and *vice versâ* when the sign is different.

34. *To find the coordinates of the intersection of three planes.*

This is only to solve three equations of the first degree for three unknown quantities (*Higher Algebra*, Art. 29, or Burnside and Panton, Art. 144). The values of the coordinates will become infinite if the determinant $(AB'C'')$ vanishes, or

$$A(B'C'' - B''C') + A'(B''C - BC'') + A''(BC' - B'C) = 0.$$

This then is the condition that the three planes should be parallel to the same line. For in such a case the line of

intersection of any two would be also parallel to this line, and could not meet the third plane at any finite distance.

35. *To find the condition that four planes should meet in a point.*

This is evidently obtained by eliminating x, y, z between the equations of the four planes, and is therefore the determinant $(AB'C''D''')$, or

$$\begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0.$$

36. *To find the volume of the tetrahedron whose vertices are any four given points.*

If we multiply the area of the triangle formed by three points, by the perpendicular on their plane from the fourth, we obtain three times the volume. The length of the perpendicular on the plane, whose equation is given (Art. 31), is found by substituting in that equation the coordinates of the fourth point, and dividing by the square root of the sum of the squares of the coefficients of x, y, z. But (Art. 32) that square root is double the area of the triangle formed by the three points. Hence *six times the volume of the tetrahedron in question is equal to the determinant*

$$\begin{vmatrix} x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & z''', & 1 \\ x'''', & y'''', & z'''', & 1 \end{vmatrix}.$$

Ex. The volume of the tetrahedron formed by four planes, whose equations are given, can be found by forming the coordinates of its angular points, and then substituting in the formula given above. The result is (*Higher Algebra*, Art. 30, or Burnside and Panton, Art. 146), that six times the volume is equal to

$$\frac{R^3}{(AB'C'')\,(A'B''C''')\,(A''B'''C)\,(A'''BC')}$$

where R is the determinant $(AB'C''D''')$, Art. 35, and the factors in the denominator express the conditions (Art. 34) that any three of the planes should be parallel to the same line.

37. It is evident, as in Plane Geometry (see *Conics*, Art. 40), that if S, S', S'' represent any three surfaces, then $aS+bS'$, where a and b are any constants, represents a surface passing through the line of intersection of S and S'; and that $aS+bS'+cS''$ represents a surface passing through the points of intersection of S, S', and S''. Thus then, if L, M, N denote any three planes, $aL+bM$ denotes a plane passing through the line of intersection of the first two, and $aL+bM+cN$ denotes a plane passing through the point common to all three.* As a particular case of the preceding $aL+b$ denotes a plane parallel to L, and $aL+bM+c$ denotes a plane parallel to the intersection of L and M (see Art. 30).

So again, four planes L, M, N, P will pass through the same point if their equations are connected by an identical relation

$$aL+bM+cN+dP\equiv 0,$$

for then any coordinates which satisfy the first three must satisfy the fourth. Conversely, given any four planes intersecting in a common point, it is easy to obtain such an identical relation. For multiply the first equation by the determinant $(A'B''C''')$, the second by $-(A''B'''C)$, the third by $(A'''BC')$, and the fourth by $-(AB'C'')$, and add, then (*Higher Algebra*, Art. 7, or Burnside and Panton, Art. 145) the coefficients of x, y, z vanish identically; and the remaining term is the determinant which vanishes (Art. 35), because the planes meet in a point. Their equations are therefore connected by the identical relation

$$L\ (A'B''C''')-M\ (A''B'''C)+N\ (A'''BC')-P\ (AB'C'')\equiv 0.$$

[This equation may be obtained by eliminating 1, x, y, z, from the four identical equations of the form $L-D\equiv Ax+By+Cz$, and equating to zero the determinant of Art. 35.

By the same method it follows that the equations of *any*

* German writers distinguish the system of planes having a line common by the name Büschel from the system having only one point common, which they call Bündel. [The corresponding English terms are generally *pencil* and *sheaf*.]

four planes are connected by an identical relation $aL+bM+cN+dP+e\equiv 0$.]

38. Given any four planes L, M, N, P not meeting in a point, it is easy to see (as at *Conics*, Art. 60) that the equation of any other plane can be thrown into the form

$$aL+bM+cN+dP=0.$$

[This relation may be found by eliminating x, y, z, 1, k from the five equations

$$\lambda x+\mu y+\nu z+\rho\equiv kA,$$
$$Ax+By+Cz+D\equiv kL, \text{ \&c.},$$

where $A\equiv\lambda x+\mu y+\nu z+\rho=0$ is the plane.

Thus the equations of any *five* planes are connected by a linear homogeneous relation.]

In general the equation of any surface of the degree n can be expressed by a homogeneous equation of the degree n between L, M, N, P (see *Conics*, Art. 289). For the number of terms in the *complete* equation of the degree n between *three* variables is the same as the number of terms in the *homogeneous* equation of the degree n between *four* variables.

Accordingly, in what follows, we shall use these *quadriplanar* coordinates, whenever by so doing our equations can be materially simplified; that is, we shall represent the equation of a surface by a homogeneous equation between four coordinates x, y, z, w; where these may be considered as denoting the perpendicular distances, or quantities proportional to the perpendicular distances (or to given multiples of the perpendicular distances) of the point from four given planes $x=0$, $y=0$, $z=0$, $w=0$.

It is at once apparent that, as in *Conics*, Art. 70, there is also a second system of interpretation of our equations, in which an equation of the first degree represents a point, and the variables are the coordinates of a plane. In fact, if $L'M'N'P'$ denote the coordinates of a fixed point, the above plane passes through it if $aL'+bM'+cN'+dP'=0$, and the coordinates of any plane through this point are subject only to this relation. The quantities, a, b, c, d may be considered

as denoting the perpendicular distances, or quantities proportional to the perpendicular distances (or to given multiples of the perpendicular distances) of the plane from four given points $a=0$, $b=0$, $c=0$, $d=0$. [For the point whose equation in plane coordinates is $a=0$ is the intersection of the three planes whose equations in point coordinates are $M=0$, $N=0$, $P=0$; if we substitute the coordinates (x_1, y_1, z_1) of this point in $aL+bM+cN+dP$ we get aL_1, which must be proportional to the perpendicular from x_1, y_1, z_1 on the plane $aL+bM+cN+dP=0$; hence a is proportional to this perpendicular since L_1 is constant.]

Ex. 1. To find the equation of the plane passing through $x'y'z'$, and through the intersection of the planes.

$Ax + By + Cz + D$, $A'x + B'y + C'z + D'$ (see *Conics*, Art. 40, Ex. 3).

Ans. $(A'x' + B'y' + C'z' + D')(Ax + By + Cz + D)$
$$= (Ax' + By' + Cz' + D)(A'x + B'y + C'z + D').$$

Ex. 2. Find the equation of the plane passing through the points ABC, figure, p. 2.

The equations of the line BC are evidently $\frac{x}{a} = 1$, $\frac{y}{b} + \frac{z}{c} = 1$. Hence obviously the equation of the required plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2$, since this passes through each of the three lines joining the three given points.

Ex. 3. Find the equation of the plane PEF in the same figure.

The equations of the line EF are $x = 0$, $\frac{y}{b} + \frac{z}{c} = 1$; and forming as above the equation of the plane joining this line to the point abc, we get

$$\frac{y}{b} + \frac{z}{c} - \frac{x}{a} = 1.$$

39. *If four planes which intersect in a right line be met by any plane, the anharmonic ratio of the pencil so formed will be constant.* For we could by transformation of coordinates make the transverse plane the plane of xy, and we should then obtain the equations of the intersections of the four planes with this plane by making $z=0$ in the equations. The resulting equations will be of the form $aL+M$, $bL+M$, $cL+M$, $dL+M$ (each equated to zero) whose anharmonic ratio (see *Conics*, Art. 59) depends solely on the constants a, b, c, d; and does not alter when by transformation of coordinates L and M come to represent different lines. [This anharmonic

ratio is evidently the same as the anharmonic ratio of the four points in which any line meets the planes.]

40. [The homogeneous coordinates (x, y, z, w) of a point are linear functions of its Cartesian coordinates and are either proportional to the perpendiculars from the point on the faces of the tetrahedron or are proportional to given multiples of these perpendiculars, but in the following argument we shall give the coordinates of a point definite values by taking them to be equal to (and not merely proportional to) the perpendiculars or to the given multiples of the perpendiculars. If we transform from the homogeneous coordinates (x, y, z, w) to another set $(\xi, \eta, \zeta, \omega)$ we must have equations of the form

$$\begin{aligned} x &= \lambda_1\xi + \lambda_2\eta + \lambda_3\zeta + \lambda_4\omega, \\ y &= \mu_1\xi + \mu_2\eta + \mu_3\zeta + \mu_4\omega, \\ z &= \nu_1\xi + \nu_2\eta + \nu_3\zeta + \nu_4\omega, \\ w &= \rho_1\xi + \rho_2\eta + \rho_3\zeta + \rho_4\omega. \end{aligned}$$

Let the coordinates of the vertices (P_1, P_2, P_3, P_4) of the second tetrahedron of reference referred to the first system be x_1, y_1, z_1, w_1; x_2, y_2, z_2, w_2, &c.; let the coordinates of P_1, P_2, P_3, P_4, referred to the second system be ξ_0, 0, 0, 0; 0, η_0, 0, 0; 0, 0, ζ_0, 0; 0, 0, 0, ω_0. Then we have $x_1 = \lambda_1\xi_0$, $y_1 = \mu_1\xi_0$, with similar equations. Hence

$$\lambda_1 = \frac{x_1}{\xi_0},\ \mu_1 = \frac{y_1}{\xi_0},\ \nu_1 = \frac{z_1}{\xi_0},\ = \rho_1\frac{w_1}{\xi_0}.]$$

Hence $$x = \frac{\xi x_1}{\xi_0} + \frac{\eta x_2}{\eta_0} + \frac{\zeta x_3}{\zeta_0} + \frac{\omega x_4}{\omega_0}.$$

Evidently similar equations give the coordinates of P referred to the other planes of reference; for instance,

$$y = \frac{\xi y_1}{\xi_0} + \frac{\eta y_2}{\eta_0} + \frac{\zeta y_3}{\zeta_0} + \frac{\omega y_4}{\omega_0},\ \&c.$$

Thus, writing down these four equations, we have the full system requisite for a transformation of coordinates from the old planes of x, y, z, w to the planes ξ, η, ζ, ω.

It will sometimes be convenient to use a single letter for $\xi : \xi_0$, &c., whereby our expressions will gain in compactness, but at the expense of apparent homogeneity.

It is evident that the transformation of coordinates is quite similar for the coordinates of planes.

41. If we denote by αx_0, βy_0, γz_0, δw_0 the perpendiculars from the vertices on the opposite sides of the original tetrahedron, α, β, γ, δ being constants, we have obviously, if A, B, C, D be the areas of those faces,

$$A\alpha x_0 = B\beta y_0 = C\gamma z_0 = D\delta w_0 = 3V$$

$$A\alpha x + B\beta y + C\gamma z + D\delta w = 3V$$

where V denotes the volume of that tetrahedron.

Thus the relation which can at once be written down by equating the volume of the tetrahedron of reference to the sum of the four tetrahedra which its faces subtend at any point, viz. $A\alpha x + B\beta y + C\gamma z + D\delta w = 3V$ may be written

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} + \frac{w}{w_0} = 1,$$

and in like manner we have

$$\frac{\xi}{\xi_0} + \frac{\eta}{\eta_0} + \frac{\zeta}{\zeta_0} + \frac{\omega}{\omega_0} = 1$$

as the relations connecting in each system the homogeneous coordinates with an absolute numerical quantity (cf. *Conics*, Art. 63).

Ex. To express the volume of a tetrahedron by the homogeneous coordinates of its vertices [when these coordinates represent actual perpendiculars on the faces of the tetrahedron of reference.]

If we multiply the determinant expression, found Art. 36, for six times the volume W by

$$\begin{vmatrix} \cos\alpha, & \cos\beta, & \cos\gamma, & 0 \\ \cos\alpha', & \cos\beta', & \cos\gamma', & 0 \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'', & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix},$$

which is the same as the determinant in Ex. 1, Art. 32, and as that of the transformation (G) Art. 18, we find

$$\begin{vmatrix} X', & Y', & Z', & 1 \\ X'', & Y'', & Z'', & 1 \\ X''', & Y''', & Z''', & 1 \\ X^{iv}, & Y^{iv}, & Z^{iv}, & 1 \end{vmatrix}$$

as the product of six times the volume W by the quantity which we may call the sine of the solid angle XYZ (Art. 56).

Now these coordinates are measured along the axes, and we want to refer

to perpendiculars on the coordinate planes. Hence we may write the new coordinates $x = X \sin p$, $y = Y \sin q$, $z = Z \sin r$, where p, q, r are the angles the axes of X, Y, Z make with the planes YZ, &c.; therefore

$$\begin{vmatrix} x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & z''', & 1 \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & 1 \end{vmatrix} = 6W \sin p \sin q \sin r \sin (XYZ),$$

or by the relations

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} + \frac{w}{w_0} = 1, \text{ \&c.},$$

$$\begin{vmatrix} x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \\ x''', & y''', & z''', & w''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & w^{\text{iv}} \end{vmatrix} = 6Ww_0 \sin p \sin q \sin r \sin (XYZ).$$

We may give this another form by remarking that the determinant reduces for the tetrahedron of reference to the continued product, which is its leading term, hence

$$x_0 y_0 z_0 w_0 = 6Vw_0 \sin p \sin q \sin r \sin (XYZ),$$

whence, dividing the former equation by this,

$$\frac{(x'y''z'''w^{\text{iv}})}{x_0 y_0 z_0 w_0} = \frac{W}{V}.$$

[If x, y, z, w are *any* homogeneous coordinates—that is, any linear functions of Cartesian coordinates or of another set of homogeneous coordinates—then it is still true that the volume of the tetrahedron is proportional to the determinant $(x'y''z'''w^{\text{iv}})$. Hence the above result still holds good.]

The Right Line.

42. The equations of any two planes taken together will represent their line of intersection, which will include all the points whose coordinates satisfy *both* equations. By eliminating x and y alternately between the equations we reduce them to a form commonly used, viz.

$$x = mz + a, \quad y = nz + b.$$

The first represents the projection of the line on the plane of xz and the second that on the plane of yz. The reader will observe that *the equations of a right line include four independent constants.*

We might form independently the equations of the line joining two points; for taking the values given (Art. 8) of the coordinates of any point on that line, solving for the ratio $m : l$ from each of the three equations there given, and equating results, we get

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''}=\frac{z-z'}{z'-z''},$$

for the required equations of the line. It thus appears that the equations of the projections of the line are the same as the equations of the lines joining the projections of two points on the line, as is otherwise evident.

43. Two right lines in space will in general not intersect. If the first line be represented by any two equations $L=0$, $M=0$, and the second by any other two $N=0$, $P=0$, then if the two lines meet in a point, each of these four planes must pass through that point, and the condition that the lines should intersect is the same as that already given (Art. 35).

Two intersecting lines determine a plane whose equation can easily be found. For we have seen (Art. 37) that when the four planes intersect, their equations satisfy an identical relation

$$aL+bM+cN+dP\equiv 0.$$

The equations therefore $aL+bM=0$, and $cN+dP=0$ must be identical and must represent the same plane. But the form of the first equation shows that this plane passes through the line L, M, and that of the second equation shows that it passes through the line N, P.

Ex. When the given lines are represented by equations of the form

$$x=mz+a,\ y=nz+b;\ x=m'z+a',\ y=n'z+b',$$

the condition that they should intersect is easily found. For solving for z from the first and third equations, and equating it to the value found by solving for the second and fourth, we get

$$\frac{a-a'}{m-m'}=\frac{b-b'}{n-n'}.$$

Again, if this condition is satisfied, the four equations are connected by the identical relation

$$(n-n')\{(x-mz-a)-(x-m'z-a')\}$$
$$=(m-m')\{(y-nz-b)-(y-n'z-b')\},$$

and therefore $\qquad (n-n')(x-mz-a)=(m-m')(y-nz-b)$

is the equation of the plane containing both lines.

44. *To find the equations of a line passing through the point $x'y'z'$, and making angles α, β, γ with the axes.*

The projections on the axes, of the distance of $x'y'z'$ from

any variable point xyz on the line, are respectively $x - x'$, $y - y'$, $z - z'$; and since these are each equal to that distance multiplied by the cosine of the angle between the line and the axis in question, we have

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma};$$

a form of writing the equations of the line which, although it includes two superfluous constants, yet on account of its symmetry between x, y, z is often used in preference to the first form in Art. 42.

Reciprocally, if we desire to find the angles made with the axes by any line, we have only to throw its equation into the form $\frac{x - x'}{A} = \frac{y - y'}{B} = \frac{z - z'}{C}$, when the direction-cosines of the line will be respectively A, B, C, each divided by the square root of the sum of the squares of these three quantities.

Ex. 1. To find the direction-cosines of $x = mz + a$, $y = nz + b$. Writing the equations in the form $\frac{x - a}{m} = \frac{y - b}{n} = \frac{z}{1}$, the direction-cosines are

$$\frac{m}{\sqrt{(1 + m^2 + n^2)}}, \frac{n}{\sqrt{(1 + m^2 + n^2)}}, \frac{1}{\sqrt{(1 + m^2 + n^2)}}.$$

Ex. 2. To find the direction-cosines of

$$\frac{x}{l} = \frac{y}{m}, z = 0. \quad Ans. \quad \frac{l}{\sqrt{(l^2 + m^2)}}, \frac{m}{\sqrt{(l^2 + m^2)}}, 0.$$

Ex. 3. To find the direction-cosines of

$$Ax + By + Cz + D = 0, A'x + B'y + C'z + D' = 0.$$

Eliminating y and z alternately we reduce to the preceding form, and the direction-cosines are $\frac{BC' - B'C}{R}$, $\frac{CA' - C'A}{R}$, $\frac{AB' - A'B}{R}$, where R^2 is the sum of the squares of the three numerators.

Ex. 4. To find the equation of the plane through the two intersecting lines

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma}; \quad \frac{x - x'}{\cos \alpha'} = \frac{y - y'}{\cos \beta'} = \frac{z - z'}{\cos \gamma'}.$$

The required plane passes through $x'y'z'$ and its perpendicular is perpendicular to two lines whose direction-cosines are given; therefore (Art. 15), the required equation is

$$(x - x')(\cos \beta \cos \gamma' - \cos \gamma \cos \beta') + (y - y')(\cos \gamma \cos \alpha' - \cos \gamma' \cos \alpha)$$
$$+ (z - z')(\cos \alpha \cos \beta' - \cos \alpha' \cos \beta) = 0.$$

Ex. 5. To find the equation of the plane passing through the two parallel lines

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma}; \quad \frac{x - x''}{\cos \alpha} = \frac{y - y''}{\cos \beta} = \frac{z - z''}{\cos \gamma}.$$

The required plane contains the line joining the given points, whose direction-cosines are proportional to $x' - x''$, $y' - y''$, $z' - z''$, the direction-cosines of the perpendicular to the plane are therefore proportional to

$$(y' - y'')\cos\gamma - (z' - z'')\cos\beta,\ (z' - z'')\cos\alpha - (x' - x'')\cos\gamma,$$
$$(x' - x'')\cos\beta - (y' - y'')\cos\alpha$$

These may therefore be taken as the coefficients of x, y, z in the required equation, while the absolute term determined by substituting $x'y'z'$ for xyz in the equation is

$$(y'z'' - y''z')\cos\alpha + (z'x'' - z''x')\cos\beta + (x'y'' - x''y')\cos\gamma.$$

[Ex. 6 Prove that the condition that the two lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1},\quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

may intersect is

$$\begin{vmatrix} x_1, & x_2, & l_1, & l_2 \\ y_1, & y_2, & m_1, & m_2 \\ z_1, & z_2, & n_1, & n_2 \\ 1, & 1, & 0, & 0 \end{vmatrix} = 0.]$$

45 *To find the equations of the perpendicular from $x'y'z'$ on the plane $Ax + By + Cz + D$* The direction-cosines of the perpendicular on the plane (Art 28) are proportional to A, B, C, hence the equations required are

$$\frac{x - x'}{A} = \frac{y - y'}{B} = \frac{z - z'}{C}.$$

46 *To find the direction-cosines of the bisector of the angle between two given lines*

As we are only concerned with *directions* it is of course sufficient to consider lines through the origin. If we take points $x'y'z'$, $x''y''z''$ one on each line, equidistant from the origin, then the middle point of the line joining these points is evidently a point on the bisector, whose equation therefore is

$$\frac{x}{x' + x''} = \frac{y}{y' + y''} = \frac{z}{z' + z''},$$

and whose direction-cosines are therefore proportional to

$$x' + x'',\ y' + y'',\ z' + z'';$$

but since x', y', z'; x'', y'', z'' are evidently proportional to the direction-cosines of the given lines, the direction-cosines of the bisector are

$$\cos\alpha' + \cos\alpha'',\ \cos\beta' + \cos\beta'',\ \cos\gamma' + \cos\gamma'',$$

each divided by the square root of the sum of the squares of these three quantities.

The bisector of the supplemental angle between the lines is got by substituting for the point $x''y''z''$ a point equidistant from the origin measured in the opposite direction, whose coordinates are $-x'', -y'', -z''$; and therefore the direction-cosines of this bisector are

$$\cos\alpha' - \cos\alpha'',\ \cos\beta' - \cos\beta'',\ \cos\gamma' - \cos\gamma'',$$

each divided by the square root of the sum of the squares of these three quantities. The square roots in question are obviously $\sqrt{(2 \pm 2\cos\delta)}$; that is, $2\cos\frac{1}{2}\delta$ and $2\sin\frac{1}{2}\delta$, if δ is the angle between the two lines.

Ex 1. The equation of the *plane* bisecting the angle between two given *planes* is found precisely as at *Conics*, Art. 35, and is

$$(x\cos\alpha + y\cos\beta + z\cos\gamma - p) \pm (x\cos\alpha' + y\cos\beta' + z\cos\gamma' - p').$$

[Ex. 2 Find the equation of the plane passing through the line of intersection of two planes and making angles θ and θ' with these planes, where

$$\frac{\sin\theta}{\sin\theta'} = k.$$

Ex. 3. Find the equation and length of the perpendicular from a point $x'y'z$ on the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.]$$

47. *To find the angle made with each other by two lines*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n};\quad \frac{x-a}{l'} = \frac{y-b}{m'} = \frac{z-c}{n'}.$$

Evidently (Arts. 13, 44),

$$\cos\theta = \frac{ll' + mm' + nn'}{\sqrt{(l^2+m^2+n^2)}\sqrt{(l'^2+m'^2+n'^2)}}.$$

COR. The lines are at right angles to each other if

$$ll' + mm' + nn' = 0.$$

Ex. To find the angle between the lines

$$\frac{x}{2} = \frac{y}{\sqrt{(3)}} = \frac{z}{\sqrt{(2)}};\quad \frac{x}{\sqrt{(3)}} = y,\ z = 0. \quad \textit{Ans. } 30°.$$

48. *To find the angle between the plane $Ax + By + Cz + D$, and the line* $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$.

The angle between the line and the plane is the comple-

ment of the angle between the line and the perpendicular on the plane, and we have therefore

$$\sin\theta = \frac{Al + Bm + Cn}{\sqrt{(l^2+m^2+n^2)}\sqrt{(A^2+B^2+C^2)}}.$$

COR. When $Al + Bm + Cn = 0$, the line is parallel to the plane, for it is then perpendicular to a perpendicular on the plane.

49. *To find the conditions that a line* $x = mz + a$, $y = nz + b$ *should be altogether in a plane* $Ax + By + Cz + D = 0$. Substitute for x and y in the equation of the plane, and solve for z, when we have

$$z = -\frac{Aa + Bb + D}{Am + Bn + C},$$

and if both numerator and denominator vanish, the value of z is indeterminate and the line is altogether in the plane. We have just seen that the vanishing of the denominator expresses the condition that the line should be parallel to the plane; while the vanishing of the numerator expresses that one of the points of the line is *in* the plane, viz. the point ab where the line meets the plane of xy.

In like manner, in order to find the conditions that a right line should lie altogether in any surface, we should substitute for x and y in the equation of the surface, and then equate to zero the coefficient of *every* power of z in the resulting equation. It is plain that the number of conditions thus resulting is one more than the degree of the surface.

[To find the condition that the line $\frac{x-x'}{\alpha} = \frac{y-y'}{\beta} = \frac{z-z'}{\gamma}$ may be altogether on a given surface, we substitute $x' + \alpha\rho$, $y' + \beta\rho$, $z' + \gamma\rho$ for x, y, z in the equation of the surface and equate to zero the coefficients of powers of ρ and the absolute term.]

Since the equations of a right line contain four constants, a right line can be determined which shall satisfy any four conditions. Hence any surface of the second degree must contain an infinity of right lines, since we have only three conditions to satisfy and have four constants at our disposal. Every surface of the third degree must contain a finite number of right lines, since the number of conditions to be satisfied is equal to the number of disposable constants. A surface of higher degree will not necessarily contain any right line lying altogether in the surface.

50. *To find the equation of the plane drawn through a given line perpendicular to a given plane.*

Let the line be given by the equations

$$Ax+By+Cz+D=0,\quad A'x+B'y+C'z+D'=0,$$

and let the plane be

$$A''x+B''y+C''z+D''=0.$$

Then any plane through the line will be of the form

$$\lambda(Ax+By+Cz+D)+\mu(A'x+B'y+C'z+D')=0,$$

and, in order that it should be perpendicular to the plane, we must have

$$(\lambda A+\mu A')A''+(\lambda B+\mu B')B''+(\lambda C+\mu C')C''=0.$$

This equation determines $\lambda:\mu$, and the equation of the required plane is

$$(A'A''+B'B''+C'C'')(Ax+By+Cz+D)$$
$$=(AA''+BB''+CC'')(A'x+B'y+C'z+D').$$

[When the equations of the given plane and line are given in the forms

$$\alpha x+\beta y+\gamma z+\delta=0,\quad \frac{x-x'}{\alpha}=\frac{y-y'}{\beta}=\frac{z-z'}{\gamma'}=0$$

the equation of the required plane is found by eliminating the constants A, B, C, D from the four equations

$$\begin{aligned} Ax+By+Cz+D&=0\\ Ax'+By'+Cz'+D&=0\\ A\alpha+B\beta+C\gamma&=0\\ A\alpha'+B\beta'+C\gamma'&=0 \end{aligned}$$

the second equation expressing that the plane contains x', y', z', the third that it is perpendicular to the given plane, and the fourth that the perpendicular to the sought plane is perpendicular to the given line. Hence the equation of the required plane is

$$\begin{vmatrix} x, & y, & z, & 1\\ x', & y', & z', & 1\\ \alpha, & \beta, & \gamma, & 0\\ \alpha', & \beta', & \gamma', & 0 \end{vmatrix}=0.]$$

51. *Given two lines, to find the equation of a plane drawn through either parallel to the other.*

First, let the given lines be the intersections of the planes L, M; N, P, whose equations are given in the most general form. Then proceeding exactly as in Art. 37, we obtain the identical relation

$$L(A'B''C''') - M(A''B'''C) + N(A'''BC') - P(AB'C'') = (A'B''C'''D),$$

the right-hand side of the equation being the determinant, whose vanishing expresses that the four planes meet in a point. It is evident then that the equations

$$L(A'B''C''') - M(A''B'''C) = 0,\ N(A'''BC') - P(AB'C'') = 0$$

represent parallel planes, since they only differ by a constant quantity; but these planes pass each through one of the given lines.

Secondly, let the lines be given by equations of the form

$$\frac{x-x'}{\cos\alpha} = \frac{y-y'}{\cos\beta} = \frac{z-z'}{\cos\gamma};\ \frac{x-x''}{\cos\alpha'} = \frac{y-y''}{\cos\beta'} = \frac{z-z''}{\cos\gamma'}.$$

Then since a perpendicular to the sought plane is perpendicular to the direction of each of the given lines, its direction-cosines may be written down by the method given in Art. 15, and the equations of the sought parallel planes are

$$(x-x')(\cos\beta\cos\gamma' - \cos\beta'\cos\gamma) + (y-y')(\cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha) + (z-z')(\cos\alpha\cos\beta' - \cos\alpha'\cos\beta) = 0,$$

$$(x-x'')(\cos\beta\cos\gamma' - \cos\beta'\cos\gamma) + (y-y'')(\cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha) + (z-z'')(\cos\alpha\cos\beta' - \cos\alpha'\cos\beta) = 0.$$

[The first of these equations, and the second similarly, may also be expressed as the determinant resulting from the elimination of λ, μ, ν, ρ from the four equations

$$\lambda x + \mu y + \nu z + \rho = 0$$
$$\lambda x' + \mu y' + \nu z' + \rho = 0$$
$$\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma = 0$$
$$\lambda\cos\alpha' + \mu\cos\beta' + \nu\cos\gamma' = 0,$$

the plane sought being $\lambda x + \mu y + \nu z + \rho = 0$.]

The *perpendicular distance between two parallel planes* is equal to the difference between the perpendiculars let fall on them from the origin, and is therefore equal to the difference between their absolute terms, divided by the square

root of the sum of the squares of the common coefficients of x, y, z. Thus the perpendicular distance between the planes last found is

$(x'-x'')(\cos\beta\cos\gamma'-\cos\beta'\cos\gamma)+(y'-y'')(\cos\gamma\cos\alpha'$
$\quad -\cos\gamma'\cos\alpha)+(z'-z'')(\cos\alpha\cos\beta'-\cos\alpha'\cos\beta)$ divided by $\sin\theta$,

where θ (see Art. 14) is the angle between the directions of the given lines. It is evident that the perpendicular distance here found is shorter than any other line which can be drawn from any point of the one plane to any point of the other.

52. *To find the equations and the magnitude of the shortest distance between two non-intersecting lines.*

The shortest distance between two lines is a line perpendicular to both, which can be found as follows: Draw through each of the lines, by Art. 50, a plane perpendicular to either of the parallel planes determined by Art. 51; then the intersection of the two planes so drawn will be perpendicular to the parallel planes, and therefore to the given lines which lie in these planes. From the construction it is evident that the line so determined meets both the given lines. Its magnitude is plainly that determined in the last article. Calculating by Art. 50 the equation of a plane passing through a line whose direction-angles are α, β, γ, and perpendicular to a plane whose direction-cosines are proportional to

$$\cos\beta'\cos\gamma-\cos\beta\cos\gamma',\ \cos\gamma'\cos\alpha-\cos\gamma\cos\alpha',\ \cos\alpha'\cos\beta - \cos\alpha\cos\beta',$$

we find that the line sought is the intersection of the two planes

$$(x-x')(\cos\alpha'-\cos\theta\cos\alpha)+(y-y')(\cos\beta'-\cos\theta\cos\beta) + (z-z')(\cos\gamma'-\cos\theta\cos\gamma)=0,$$

$$(x-x'')(\cos\alpha-\cos\theta\cos\alpha')+(y-y'')(\cos\beta-\cos\theta\cos\beta') + (z-z'')(\cos\gamma-\cos\theta\cos\gamma')=0.$$

The direction-cosines of the shortest distance must plainly be proportional to

$$\cos\beta'\cos\gamma-\cos\beta\cos\gamma',\ \cos\gamma'\cos\alpha-\cos\gamma\cos\alpha',\ \cos\alpha'\cos\beta - \cos\alpha\cos\beta'.$$

Ex. To find the shortest distance δ between the right line

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p,$$
$$x\cos\alpha' + y\cos\beta' + z\cos\gamma' = p',$$

and that joining the points $P'(x', y', z')$ and $P''(x''y''z'')$.

Denoting by L, M the perpendiculars from any point xyz on the two given planes and by $L'M'$, $L''M''$ those from the points P', P''; $L + \lambda M = 0$ is the equation of any plane passing through the first right line. [If it is parallel to the line $P'P''$, whose direction-cosines are proportional to $x' - x''$, $y' - y''$, $z' - z''$ then (Art. 48, Cor.)

$$\lambda(M' - M'') + L' - L'' = 0.$$

Hence $LM' - L'M + L''M - LM'' = 0$ is the required plane through LM.

The equation of a parallel plane through $P'P''$ differs from this equation by a constant and we determine this constant by expressing the condition that it passes through P' or P''. Hence the equation of the second plane is

$$LM' - L'M + L''M - LM'' + L'M'' - L''M' = 0.]$$

Thus by dividing $L'M'' - L''M'$ by the square root of the sum of squares of coefficients of x, y and z in either of these equations, we find the required shortest distance.

The result of reducing this expression can also be arrived at thus: $L'M'$ are the lengths of perpendiculars from P' on the two given planes. They are both contained in a plane through P' at right angles to the right line LM. In like manner $L''M''$ are contained in a parallel plane through P''. Now considering projections on either of these planes, if ϕ be the angle between the planes L and M, double the area of the triangle subtended by the projection of $P'P''$ at the intersection of L, M multiplied by $\sin\phi = L'M'' - L''M'$. But that double area is evidently the product of the required shortest distance δ between the two given lines by the projection of $P'P''$. Hence, calling θ the angle between the two lines, we see that

$$L'M'' - L''M' = (P'P'') \,.\, \delta \,.\, \sin\theta \sin\phi.$$

The Six Coordinates of a Right Line.*

53. When the equations of a right line are written in the form $\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n}$ to any system of coordinate axes, they appear to involve five independent quantities, viz. x', y', z', and the ratios $l : m : n$. But it is easily seen that x', y', z', occur in groups which are not independent, and the total number of independent constants is only four, as we saw in Art. 42. In fact, if we denote respectively by a, b, c the quantities

* Sections 53-53e may be omitted on first reading.

$mz' - ny'$, $nx' - lz'$, $ly' - mx'$, we have at once the relation $la + mb + nc = 0$, and subject to this the equations of the right line are any two of the four equations

$$\begin{array}{rrrl} & ny & - mz + a & = 0, \\ -nx & & + lz + b & = 0, \\ mx & - ly & + c & = 0, \\ ax & + by & + cz & = 0, \end{array}$$

for by the above relation the remaining two can in all cases be deduced.

We have now six quantities a, b, c, l, m, n which serve to determine the position of a right line provided the relation $la + mb + nc = 0$ hold, and these we shall call the *six coordinates of the right line.*

If we examine the conditions, as in Art. 49, that this right line may be wholly contained in the plane

$$Ax + By + Cz + D = 0,$$

we find they are any two of the four equations

$$\begin{array}{rrrl} & Bc & - Cb + Dl & = 0, \\ -Ac & & + Ca + Dm & = 0, \\ Ab & - Ba & + Dn & = 0, \\ Al & + Bm & + Cn & = 0, \end{array}$$

from which also by the universal relation $al + bm + cn = 0$, the remaining two can in all cases be deduced. It is important to observe that the quantities a, b, c which are the functions $mz - ny$, $nx - lz$, $ly - mx$ of the coordinates x, y, z of *any* point on the right line have the same values for each point on it. We are thus enabled to express in x, y, z coordinates the relation equivalent to any given relation in a, b, c. Again, if we suppose the x, y, z axes rectangular, and that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, it is easily seen, by Art. 15, that a, b, c are the coordinates of a point on the perpendicular through the origin to the plane passing through the origin and the given line, and at a distance from the origin equal to that of the given line.

[The coordinates of the line joining the points x', y', z' to x'', y'', z'', are

$$y'z'' - y''z',\ z'x'' - z''x',\ x'y'' - x''y',\ x' - x'',\ y' - y'',\ z' - z''.$$

If the components, along the axis, of a force acting along the line are X, Y, Z,

and the moments of this force round the axis L, M, N, then—with the usual convention for the signs of these moments—the coordinates of the line are proportional to $-L$, $-M$, $-N$, X, Y, Z. For $X : Y : Z :: x' - x'' : y' - y'' : z' - z''$, and $L : M : N :: y'Z - z'Y : z'X - x'Z : x'Y - y'X$. The relation $LX + MX + NZ = 0$ expresses that the system of forces reduces to a single force.

If the lines are given as the intersections of two planes $Ax + \&c. = 0$ and $A'x + \&c. = 0$, we find the coordinates by eliminating xyz in turn from the equations of the two planes: we then get

$$l = BC' - B'C,\ m = CA' - C'A,\ n = AB' - A'B,$$
$$a = AD' - A'D,\ b = BD' - B'D,\ c = CD' - C'D,$$

and these ratios evidently remain unaltered if we replace the two planes by any two planes through their line of intersection.]

Ex. To express by the coordinates of two right lines the shortest distance between them.

The expression found at the close of Art. 51 for the product of the shortest distance δ between two right lines by the sine of the angle θ at which they are inclined may be written

$$\begin{vmatrix} x' - x'', & \cos\alpha, & \cos\alpha' \\ y' - y'', & \cos\beta, & \cos\beta' \\ z' - z'', & \cos\gamma, & \cos\gamma' \end{vmatrix}.$$

If we replace $\cos\alpha$, &c., by l', &c., $\cos\alpha'$, &c., by l'', &c. this may be written

$$\begin{vmatrix} x', & l', & l'' \\ y', & m', & m'' \\ z', & n', & n'' \end{vmatrix} - \begin{vmatrix} x'', & l', & l'' \\ y'', & m', & m'' \\ z'', & n', & n'' \end{vmatrix}$$

in which we see that the coordinates of the points x', &c., occur only in the groups mentioned above.

Hence in the notation of this article, also omitting reference to sign,

$$\delta \sin\theta = l'a'' + m'b'' + n'c'' + l''a' + m''b' + n''c'.$$

This quantity has been called by Cayley (*Trans. Cambridge Phil. Soc.*, Vol. XI. part ii. 1868) the *moment* of the two lines. [It is proportional to the moment round either line of force F acting along the other.

The condition that the two lines may intersect is that this "moment" vanishes.]

53*a*. If we had employed quadriplanar coordinates in Art. 8 we should have used for the coordinates of any point P on the line joining P_1, P_2,

$$x = lx_1 + mx_2,\ y = ly_1 + my_2,\ z = lz_1 + mz_2,\ w = lw_1 + mw_2,$$

from which, by eliminating l and m, we find each determinant of the matrix

$$\left\| \begin{matrix} x, & y, & z, & w \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \end{matrix} \right\| = 0.$$

These four determinants contain the coordinates of P_1, P_2 only in the groups

$$\begin{matrix} (y_1z_2), & (z_1x_2), & (x_1y_2), \\ (x_1w_2), & (y_1w_2), & (z_1w_2), \end{matrix}$$

which are connected by the identity

$$(y_1z_2)\ (x_1w_2) + (z_1x_2)\ (y_1w_2) + (x_1y_2)\ (z_1w_2) = 0.$$

Thus these six quantities so connected amount to four independent ratios determining the equations, and are *homogeneous coordinates of the right line*; we shall frequently denote them, for brevity, by the letters

$$\begin{matrix} p, & q, & r, \\ s, & t, & u, \end{matrix}$$

with or without two suffixes to indicate, as may sometimes be required, the two points determining the right line; in all cases these quantities are subject to the relation

$$ps + qt + ru = 0.$$

The geometrical value of these coordinates [when the coordinates x, y, z, w, are equal to the actual perpendiculars] was obtained Ex. Art. 52, where we saw that each of them, as, for instance, (y_1z_2) is the product of the distance P_1P_2, by the sine of the angle between the planes which are named in it, multiplied into the shortest distance of P_1P_2 from the edge in which those planes intersect and into the sine of the angle between that edge and P_1P_2.

[If the coordinates (x, y, z, w) are proportional to constant multiples of the actual perpendiculars $(\alpha, \beta, \gamma, \delta)$, the coordinates of a line are proportional to constant multiples of the geometrical quantities just defined. For if $x = a\alpha$, $y = b\beta$, $z = c\gamma$, $w = d\delta$, where a, b, c, d are constants, then $p = bc\ (\beta\gamma' - \beta'\gamma)$, $q = ca(\gamma\alpha' - \gamma'\alpha)$, &c.

The coordinates of the line may also be expressed in terms of the coordinates of any two of the points where the line meets the faces of the tetrahedron of reference; thus $x = 0$ meets the line in the point $0, r, -q, s$; $w = 0$ meets it in s, t, u, 0.]

The equations connecting the coordinates of any point with the coordinates of any right line passing through it are any two of the four

$$\begin{aligned} yu - zt + wp &= 0, \\ -xu \quad + zs + wq &= 0, \\ xt - ys \quad + wr &= 0, \\ xp + yq + zr \quad &= 0, \end{aligned}$$

from which always by $ps + qt + ru = 0$ the remaining two can be deduced. These are the equations of a line as *locus* or *ray*.

[Ex. Putting w = a constant, we see that the coordinates of Art. 53 are special cases of the coordinates in this Art.; $w = 0$ becomes the plane at infinity.]

53*b*. In like manner, Art. 38, if $a_1b_1c_1d_1$, $a_2b_2c_2d_2$ be the coordinates of two planes Π_1, Π_2, the coordinates of any plane through their line of intersection are

$$a = \lambda a_1 + \mu a_2,\ b = \lambda b_1 + \mu b_2,\ c = \lambda c_1 + \mu c_2,\ d = \lambda d_1 + \mu d_2,$$

hence for a line regarded as *envelope* or *axis*, we have the system of equations

$$\left\| \begin{matrix} a, & b, & c, & d \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \end{matrix} \right\| = 0,$$

which, adopting a notation in analogy with what precedes,

$$(b_1c_2) = \pi_{12},\ (c_1a_2) = \kappa_{12},\ (a_1b_2) = \rho_{12},$$
$$(a_1d_2) = \sigma_{12},\ (b_1d_2) = \tau_{12},\ (c_1d_2) = \upsilon_{12},$$

may be written, omitting suffixes,

$$\begin{aligned} b\upsilon - c\tau + d\pi &= 0, \\ -a\upsilon \quad + c\sigma + d\kappa &= 0, \\ a\tau - b\sigma \quad + d\rho &= 0, \\ a\pi + b\kappa + c\rho \quad &= 0, \end{aligned}$$

subject to

$$\pi\sigma + \kappa\tau + \rho\upsilon = 0.$$

If this line contain the point P_1, since then

$$ax_1 + by_1 + cz_1 + dw_1 = 0,$$

we may substitute for a and b in terms of c and d and make the coefficients of c and d vanish; and similarly for the others, hence in this case

$$\begin{aligned} y_1\rho - z_1\kappa + w_1\sigma &= 0,\\ -x_1\rho \quad + z_1\pi + w_1\tau &= 0,\\ x_1\kappa - y_1\pi \quad + w_1\nu &= 0,\\ x_1\sigma + y_1\tau + z_1\nu \quad &= 0. \end{aligned}$$

In like manner, if in the last article we had sought for the conditions that the ray should be contained in the plane a, b, c, d, we should have found

$$\begin{aligned} br - cq + ds &= 0,\\ -ar \quad + cp + dt &= 0,\\ aq - bp \quad + du &= 0,\\ as + bt + cu \quad &= 0. \end{aligned}$$

Further, if we have the point P_2 also on the axis, we find

$$p : q : r : s : t : u = \sigma : \tau : \nu : \pi : \kappa : \rho,$$

or in full, if the line joining P_1 to P_2 be identical with the line in which Π_1, Π_2 intersect, each determinant vanishes in the matrix,

$$\left\| \begin{matrix} (y_1z_2), & (z_1x_2), & (x_1y_2), & (x_1w_2), & (y_1w_2), & (z_1w_2) \\ (a_1d_2), & (b_1d_2), & (c_1d_2), & (b_1c_2), & (c_1a_2), & (a_1b_2) \end{matrix} \right\|.$$

Thus we see, that equations in the homogeneous coordinates of a right line are capable of being expressed in either system, the passage from one to the other being effected by an interchange of the coordinates p and s, q and t, r and u.

N.B.—These results are merely another way of presenting the four simultaneous relations

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 + d_1w_1 &= 0,\\ a_1x_2 + b_1y_2 + c_1z_2 + d_1w_2 &= 0,\\ a_2x_1 + b_2y_1 + c_2z_1 + d_2w_1 &= 0,\\ a_2x_2 + b_2y_2 + c_2z_2 + d_2w_2 &= 0. \end{aligned}$$

[Ex. The six anharmonic ratios of the points where a line meets the faces of the tetrahedron of reference are the three quantities $-\dfrac{qt}{ru}$, $-\dfrac{ru}{ps}$, $-\dfrac{ps}{qt}$ and their reciprocals.

Using ray coordinates and replacing them by axial coordinates, we find that the anharmonic ratios of the planes joining these four points to the vertices of the tetrahedron are the same six quantities. Hence

The anharmonic ratio of the four points in which a line meets the faces of a tetrahedron is equal to the anharmonic ratio of the four planes joining the line to the vertices (von Staudt).]

53*c*. The determinant of the homogeneous coordinates of four points

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix},$$

whose geometric value we deduced in Ex. Art. 41, may be written out in full $p_{12}s_{34} + s_{12}p_{34} + q_{12}t_{34} + t_{12}q_{34} + r_{12}u_{34} + u_{12}r_{34}$, where $p_{mn} = y_m z_n - y_n z_m$, $s_{mn} = x_m w_n - x_n w_m$, &c.

Now when the line joining points 1 and 2 intersects the line joining 3 and 4, the four points are coplanar and the determinant vanishes.

Hence it appears that *the condition that two right lines*

$$\begin{pmatrix} p, & q, & r \\ s, & t, & u \end{pmatrix}, \begin{pmatrix} p', & q', & r' \\ s', & t', & u' \end{pmatrix}$$

should intersect is

$$ps' + sp' + qt' + tq' + ru' + ur' = 0.$$

53*d*. By what precedes we can see how *to determine the lines which meet four given right lines*. For if the coordinates of the required line be $\begin{matrix} p, & q, & r, \\ s, & t, & u, \end{matrix}$ and of the given lines $\begin{matrix} p_1, & q_1, & r_1, \\ s_1, & t_1, & u_1, \end{matrix}$ &c., we have

$$\begin{aligned} ps_1 + qt_1 + ru_1 + sp_1 + tq_1 + ur_1 &= 0, \\ ps_2 + qt_2 + ru_2 + sp_2 + tq_2 + ur_2 &= 0, \\ ps_3 + qt_3 + ru_3 + sp_3 + tq_3 + ur_3 &= 0, \\ ps_4 + qt_4 + ru_4 + sp_4 + tq_4 + ur_4 &= 0, \end{aligned}$$

which determine p, q, r, s linearly in terms of t and u, and when these values are substituted in the universal relation

$$ps + qt + ru = 0,$$

a quadratic is found in $t : u$, which determines the lines, two in number, which are required.

53*e*. In the coordinates of a line we have in transformation to consider the transformed coordinates of two points or planes. *Ex. gr.* considering

$$x = x_1X + x_2Y + x_3Z + x_4W, \quad x' = x_1X' + x_2Y' + x_3Z' + x_4W',$$
$$y = y_1X + y_2Y + y_3Z + y_4W, \quad y' = y_1X' + y_2Y' + y_3Z' + y_4W',$$

&c., we have

$$\begin{vmatrix} y, & z \\ y', & z' \end{vmatrix} = \begin{Vmatrix} y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{Vmatrix} \begin{Vmatrix} X, & Y, & Z, & W \\ X', & Y', & Z', & W' \end{Vmatrix},$$

or

$$p = p_{23}P + p_{31}Q + p_{12}R + p_{14}S + p_{24}T + p_{34}U,$$
$$q = q_{23}P + q_{31}Q + q_{12}R + q_{14}S + q_{24}T + q_{34}U,$$
$$r = r_{23}P + r_{31}Q + r_{12}R + r_{14}S + r_{24}T + r_{34}U,$$
$$s = s_{23}P + s_{31}Q + s_{12}R + s_{14}S + s_{24}T + s_{34}U,$$
$$t = t_{23}P + t_{31}Q + t_{12}R + t_{14}S + t_{24}T + t_{34}U,$$
$$u = u_{23}P + u_{31}Q + u_{12}R + u_{14}S + u_{24}T + u_{34}U,$$

the coefficients of the transformation evidently being the coordinates of the edges of the new tetrahedron referred to the old.

If we multiply these equations in order by s_{14}, t_{14}, u_{14}, p_{14}, q_{14}, r_{14} and add, we evidently solve for P in terms of the old coordinates, and (Art. 53c) the factor on P is the modulus of transformation; the coefficients of Q, R, S, T, U vanish identically.

Some Properties of Tetrahedra.

54. *To find the relation between the six lines joining any four points in a plane.*

Let a, b, c be the sides of the triangle formed by any three of them ABC, and let d, e, f be the lines joining the fourth point D to these three. Let the angles subtended at D by a, b, c be α, β, γ; then we have $\cos\alpha = \cos(\beta \pm \gamma)$, whence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1.$$

This relation will be true whatever be the position of D, either within or without the triangle ABC. But

$$\cos\alpha = \frac{e^2 + f^2 - a^2}{2ef}, \quad \cos\beta = \frac{f^2 + d^2 - b^2}{2fd}, \quad \cos\gamma = \frac{d^2 + e^2 - c^2}{2de}.$$

Substituting these values and reducing, we find for the required relation

$$a^2(d^2 - e^2)(d^2 - f^2) + b^2(e^2 - f^2)(e^2 - d^2) + c^2(f^2 - d^2)(f^2 - e^2)$$
$$+ a^2d^2(a^2 - b^2 - c^2) + b^2e^2(b^2 - c^2 - a^2) + c^2f^2(c^2 - a^2 - b^2)$$
$$+ a^2b^2c^2 = 0,$$

a relation otherwise deduced, *Conics*, p. 134.

55. *To express the volume of a tetrahedron in terms of its six edges.*

Let the sides of a triangle formed by any face ABC be a, b, c, the perpendicular on that face from the remaining vertex be p, and the distances of the foot of that perpendicular from A, B, C be d', e', f'. Then a, b, c, d', e', f' are connected by the relation given in the last article. But if d, e, f be the remaining edges $d^2 = d'^2 + p^2$, $e^2 = e'^2 + p^2$, $f^2 = f'^2 + p^2$; whence $d^2 - e^2 = d'^2 - e'^2$, &c., and putting in these values we get

$$-F = p^2\,(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4),$$

where F is the quantity on the left-hand side of the equation, in the last article. Now the quantity multiplying p^2 is 16 times the square of the area of the triangle ABC, and since p multiplied by this area is three times the volume of the pyramid, we have $F = -144\,V^2$.

[This result may also be obtained by using Art. 32, Ex. 1. For if α, β, γ, are the angles between three concurrent edges d, e, f

$$36V^2 = \begin{vmatrix} 1, & \cos\gamma, & \cos\beta \\ \cos\gamma, & 1, & \cos\alpha \\ \cos\beta, & \cos\alpha, & 1 \end{vmatrix} d^2e^2f^2 \equiv \Lambda d^2e^2f^2.$$

Using $\cos\alpha = \dfrac{e^2 + f^2 - a^2}{2ef}$ &c., we express Λ in terms of the edges. $\Lambda = 0$ if all the edges of the tetrahedron are coplanar; in fact $F = -4\Lambda d^2e^2f^2$.

Ex. 1. If V, V' are the volumes of two tetrahedra, and r_{11}, r_{12}, &c., the distances between corresponding vertices, then

$$288VV' = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & r_{11}^2, & r_{12}^2, & r_{13}^2, & r_{14}^2 \\ 1, & r_{21}^2, & r_{22}^2, & r_{23}^2, & r_{24}^2 \\ 1, & r_{31}^2, & r_{32}^2, & r_{33}^2, & r_{34}^2 \\ 1, & r_{41}^2, & r_{42}^2, & r_{43}^2, & r_{44}^2 \end{vmatrix}.$$

By identifying the tetrahedra we express V^2 as a determinant involving the six edges.

Ex. 2. The identical relation connecting the distances of five points in space is the determinant of the sixth order

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & (12)^2, & (13)^2, & (14)^2, & (15)^2 \\ 1, & (21)^2, & 0, & (23)^2, & (24)^2, & (25)^2 \\ & \&c. & & & \&c. & \end{vmatrix}.$$

(Use the method of *Conics*, p. 134.)

The same method can be used to find the identical relation connecting the "distances" between $n + 2$ points in "n-dimensional space," assuming

that the square of the distance between two points $(x_1, x_2, x_3 \ldots x_n)$ and $(x'_1, x'_2, x'_3 \ldots x'_n)$ is given by $d^2 = (x_1 - x'_1)^2 + (x_1 - x'_2)^2 + \ldots + (x_n - x'_n)^2$.]

56. *To find the relation between the six arcs joining four points on the surface of a sphere.*

We proceed precisely as in Art. 54, only substituting for the formulæ there used the corresponding formulæ for spherical triangles, and if $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ represent the *cosines* of the six arcs in question, we get

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \phi^2 - \alpha^2\delta^2 - \beta^2\epsilon^2 - \gamma^2\phi^2 + 2\alpha\beta\delta\epsilon + 2\beta\gamma\epsilon\phi + 2\gamma\alpha\delta\phi - 2\alpha\beta\gamma - 2\alpha\epsilon\phi - 2\beta\delta\phi - 2\gamma\delta\epsilon = 1.$$

This relation may be otherwise proved as follows: Let the direction-cosines of the radii to the four points be

$$\begin{array}{lll} \cos\alpha, & \cos\beta, & \cos\gamma, \\ \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\alpha''', & \cos\beta''', & \cos\gamma'''. \end{array}$$

Now from this matrix we can form (by the method of *Higher Algebra*, Art. 25, or Burnside and Panton, Art. 143) a determinant which shall vanish identically, and which (substituting $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = \cos ab$, &c.) is

$$\begin{vmatrix} 1, & \cos ab, & \cos ac, & \cos ad \\ \cos ba, & 1, & \cos bc, & \cos bd \\ \cos ca, & \cos cb, & 1, & \cos cd \\ \cos da, & \cos db, & \cos dc, & 1 \end{vmatrix} = 0,$$

which expanded has the value written above.

This relation might have been otherwise derived from the properties of tetrahedra as follows:—

Calling the areas of the four faces of a tetrahedron A, B, C, D; and denoting by AB the internal angle between the planes A and B, &c., we have evidently any face equal to the sum of the projections on it of the other three faces. Hence we can write down

$$\begin{array}{rrrrl} -A & +B\cos AB & +C\cos AC & +D\cos AD & = 0, \\ A\cos BA & -B & +C\cos BC & +D\cos BD & = 0, \\ A\cos CA & +B\cos CB & -C & +D\cos CD & = 0, \\ A\cos DA & +B\cos DB & +C\cos DC & -D & = 0, \end{array}$$

from which we can eliminate the areas A, B, C, D, and get a determinant relation between the six angles of intersection of the four planes.

Now as these are any four planes, the perpendiculars let fall on them from any point will meet a sphere described with that point as centre in four quite arbitrary points, say a, b, c, d, and each angle such as ab is the supplement of the corresponding angle AB between the planes; hence the former condition.

N.B.—The vanishing of a determinant (see *Higher Algebra*, Art. 33, Ex. 1, or Burnside and Panton, Art. 146) shows that the first minors of any one row are respectively proportional to the corresponding first minors of any other. We see by this article that the minors of the second determinant, and therefore those of the first, are proportional to the areas of the faces of the tetrahedron.

The reader will not find it difficult to show that for any four points on the sphere, each first minor of the corresponding determinant is the function mentioned in Ex. 1, Art. 32 of one of the four spherical triangles formed by the points. This function has been called by v. Staudt (*Crelle* 24, p. 252, 1842), the sine of the solid angle that the triangle subtends at the centre of the sphere. [That is to say *the areas of the faces of a tetrahedron are proportional to the "sines" of the "solid angles" formed by the perpendiculars from a point on the faces.*]

57. *To find the radius of the sphere circumscribing a tetrahedron.*

Since any side a of the tetrahedron is the chord of the arc whose cosine is α, we have $\alpha = 1 - \dfrac{a^2}{2r^2}$, with similar expressions for β, γ, &c.; and making these substitutions, the first formula of the last paragraph becomes

$$\frac{F}{4r^6} + \frac{2a^2d^2b^2e^2 + 2b^2e^2c^2f^2 + 2c^2f^2a^2d^2 - a^4d^4 - b^4e^4 - c^4f^4}{16r^8} = 0$$

whence if $$ad + be + cf = 2S,$$

we have $$r^2 = \frac{S(S-ad)(S-be)(S-cf)}{36V^2}$$

which has been otherwise deduced (*Higher Algebra*, Art. 26, Ex. 9).

[Ex. 1. r may be found directly in terms of the edges by using Art. 55, Ex. 2, putting $(15) = (25) = (35) = (45) = r$. Breaking up the determinant, r^2 is expressed as the ratio of two determinants.

Ex. 2. The equation $6rV = \sqrt{S(S-ad)(S-be)(S-cf)}$ can be proved by *inverting* from one corner of the tetrahedron without using the expression for V in terms of the edges, from the ordinary formula for the area of a triangle in terms of the sides (R. Russell).

Ex. 3. The shortest distance between two opposite edges a, d of a tetrahedron is $\dfrac{6V}{ad\sin\theta}$, where θ is the angle between the edges.

Ex. 4. $2ad\cos\theta = b^2 + e^2 - c^2 - f^2$.

Ex. 5. The six planes passing each through an edge of a tetrahedron and bisecting the opposite edge meet in a point.

Ex. 6. The six planes bisecting the angles between consecutive faces of a tetrahedron meet in a point.

Ex. 7. The six planes bisecting externally the angles between consecutive faces of a tetrahedron meet the opposite edges in six coplanar points.

Ex. 8. *If the four lines joining corresponding vertices of two tetrahedra meet in a point, the lines of intersection of the corresponding faces lie in a plane, and conversely.* This is an analogue of Desargues' theorem for perspective triangles.

Let $ABCD$ be the tetrahedron of reference, $A'B'C'D'$ any other tetrahedron. If $(\alpha, \beta, \gamma, \delta)$ is the given point, the coordinates of A' are of the form $\lambda + \alpha, \beta, \gamma, \delta$; those of B' are $\alpha, \mu + \beta, \gamma, \delta$; those of C' are $\alpha, \beta, \nu + \gamma, \delta$, and those of D' are $\alpha, \beta, \gamma, \rho + \delta$. The plane sought is then $\dfrac{x}{\lambda} + \dfrac{y}{\mu} + \dfrac{z}{\nu} + \dfrac{w}{\rho} = 0$.

Ex. 9. Distances d_1, d_2, d_3, d_4 are laid off on four lines in space, and V_{12} is the volume of the tetrahedron formed by d_1, d_2; if a transversal meets the lines in A, B, C, D prove

$$\frac{V_{23}V_{14}}{BC \cdot AD} + \frac{V_{31}V_{24}}{CA \cdot BD} + \frac{V_{12}V_{34}}{AB \cdot CD} = 0 \text{ (R. Russell).]}$$

CHAPTER IV.

PROPERTIES COMMON TO ALL SURFACES OF THE SECOND DEGREE.*

58. We shall write the general equation of the second degree.

$$(a, b, c, d, f, g, h, l, m, n)\ (x, y, z, 1)^2 = 0, \text{ or}$$
$$ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz = 0.$$

This equation contains ten terms, and since its signification is not altered, if by division we make one of the coefficients unity, it appears that nine conditions are sufficient to determine a surface of the second degree, or, as we shall call it for shortness, a *quadric*† surface. Thus, if we are given nine points on the surface, by substituting successively the coordinates of each in the general equation, we obtain nine equations which are sufficient to determine the nine unknown quantities $\frac{b}{a}$, $\frac{c}{a}$, &c. And, in like manner, the number of conditions necessary to determine a surface of the n^{th} degree is one less than the number of terms in the general equation.

The equation of a quadric may also (see Art. 38) be expressed as a homogeneous function of the equations of four given planes x, y, z, w.

$$(a, b, c, d, f, g, h, l, m, n)\ (x, y, z, w)^2 = 0 \text{ or}$$
$$ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0.$$

For the nine independent constants in the equation last written may be so determined that the surface shall pass through nine

* The reader will compare the corresponding discussion of the equation of the second degree (*Conics*, Chap. X.) and observe the identity of the methods now pursued and the similarity of many of the results obtained.

† Surfaces of the second degree are often called *conicoids*.

given points, and therefore may coincide with any given quadric. In like manner (see *Conics*, Art. 69) any ordinary x, y, z equations may be made homogeneous by the introduction of the linear unit (which we shall call w); and we shall frequently employ equations written in this form for the sake of greater symmetry in the results. We shall, however, for simplicity, commence with x, y, z coordinates.

59. The coordinates are transformed to any parallel axes drawn through a point $x'y'z'$, by writing $x+x'$, $y+y'$, $z+z'$ for x, y, z respectively (Art. 16). The result of this substitution will be that the coefficients of the highest powers of the variables (a, b, c, f, g, h) will remain unaltered, that the new absolute term will be U' (where U' is the result of substituting x', y', z' for x, y, z in the given equation), that the new coefficient of x will be $2(ax'+hy'+gz'+l)$ or $\frac{dU'}{dx'}$, and, in like manner, that the new coefficients of y and z will be $\frac{dU'}{dy'}$, and $\frac{dU'}{dz'}$. We shall find it convenient to use the abbreviations U_1, U_2, U_3 for $\frac{1}{2}\frac{dU}{dx}$, $\frac{1}{2}\frac{dU}{dy}$, $\frac{1}{2}\frac{dU}{dz}$.

60. We can transform the general equation to polar coordinates by writing $x=\lambda\rho$, $y=\mu\rho$, $z=\nu\rho$ (where, if the axes be rectangular, λ, μ, ν are equal to $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and if they are oblique (see note, p. 7) λ, μ, ν are still quantities depending only on the angles the line makes with the axes) when the equation becomes

$$\rho^2(a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu)$$
$$+2\rho(l\lambda+m\mu+n\nu)+d=0.$$

This being a quadratic gives *two* values for the length of the radius vector corresponding to any given direction; in accordance with what was proved (Art. 23), viz. that *every right line meets a quadric in two points.* It follows that *every plane cuts a quadric in a conic.*

61. Let us consider first the case where the origin is on the surface (and therefore $d=0$), in which case one of the roots of the above quadratic is $\rho=0$; and let us seek the condition that the radius vector should touch the surface at the origin. In this case obviously the second root of the quadratic will also vanish, and the required condition is therefore $l\lambda+m\mu+n\nu=0$. If we multiply by ρ and replace $\lambda\rho$, $\mu\rho$, $\nu\rho$, by x, y, z, this becomes

$$lx+my+nz=0,$$

and evidently expresses that the radius vector lies in a certain fixed plane. And since λ, μ, ν are subject to no restriction but that already written, *every* radius vector through the origin drawn in this plane touches the surface.

Hence we learn that at a given point on a quadric an infinity of tangent *lines* can be drawn, that these lie all in one plane which is called the *tangent plane* at that point; and that if the equation of the surface be written in the form $u_2+u_1=0$, then $u_1=0$ is the equation of the tangent plane at the origin.

62. We can find by transformation of coordinates the equation of the tangent plane at any point $x'y'z'$ in the surface. For when we transform to this point as origin, the absolute term vanishes, and the equation of the tangent plane is (Art. 59)

$$xU'_1+yU'_2+zU'_3=0,$$

or, transforming back to the old axes,

$$(x-x')\ U'_1+(y-y')\ U'_2+(z-z')\ U'_3=0.$$

This may be written in a more symmetrical form by the introduction of the linear unit w, when, since U is now a homogeneous function, and the point $x'y'z'w'$ is to satisfy the equation of the surface, we have

$$x'U'_1+y'U'_2+z'U'_3+w'U'_4\equiv U'=0.$$

Adding this to the equation last found, we have the equation of the tangent plane in the form

$$xU'_1+yU'_2+zU'_3+wU'_4=0;$$

or, writing at full length,

$$x(ax'+hy'+gz'+lw')+y(hx'+by'+fz'+mw')$$
$$+z(gx'+fy'+cz'+nw')+w(lx'+my'+nz'+dw')=0.$$

This equation, it will be observed, is symmetrical between $xyzw$ and $x'y'z'w'$, and may likewise be written

$$x'U_1+y'U_2+z'U_3+w'U_4=0.$$

[Homogeneous coordinates, in the most general sense, are linear functions of oblique or rectangular Cartesian coordinates $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$. Hence, p and p' being quite arbitrary

$$\begin{aligned} p\xi &= \lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4 w \\ p\eta &= \mu_1 x + \mu_2 y + \mu_3 z + \mu_4 w \text{ \&c., and} \\ p'\xi' &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z' + \lambda_4 w' \\ p'\eta' &= \mu_1 x' + \mu_2 y' + \mu_3 z' + \mu_4 w' \text{ \&c.} \end{aligned}$$

If we transform $x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} + w'\frac{dU}{dw}$

by using $\frac{dU}{dx} = \frac{d\phi}{d\xi}\cdot\frac{d\xi}{dx} + \frac{d\phi}{d\eta}\cdot\frac{d\eta}{dx} + \frac{d\phi}{d\zeta}\cdot\frac{d\zeta}{dx} + \frac{d\phi}{d\omega}\cdot\frac{d\omega}{dx}$ and similar equations, where $\phi = 0$ is the equation $U = 0$ expressed in ξ, η, ζ, ω, we easily find that the result, neglecting the arbitrary multiples p, p', is $\xi'\frac{d\phi}{d\xi} + \eta'\frac{d\phi}{d\eta} + \zeta'\frac{d\phi}{d\zeta} + \omega'\frac{d\phi}{d\omega}$.

We have thus proved that the equation of the tangent plane is deduced in the same way in whatever system of homogeneous coordinates the equation of the surface may be expressed.]

63. *To find the equation connecting the coordinates of the point of contact $xyzw$ of a tangent plane or line with the coordinates of any fixed point $x'y'z'w'$ on the plane or line.*

The equation last found expresses a relation between $xyzw$, the coordinates of any point on the tangent plane, and $x'y'z'w'$ its point of contact; and since now we wish to indicate that the former coordinates are given and the latter sought, we have only to remove the accents from the latter and accentuate the former coordinates, when we find that the point of contact must lie in the plane

$$xU_1'+yU_2'+zU_3'+wU_4'=0,$$

which is called the *polar plane* of the given point, and the point is called the pole of the plane. Since the point of contact need satisfy no other condition, the tangent plane at *any* of the points where the polar plane meets the surface will pass through the given point; and the line joining that point of contact to the given point will be a tangent line to the surface. If all the points of intersection of the polar plane and the surface be joined to the given point, we shall

have all the lines which can be drawn through that point to touch the surface, and the assemblage of these lines forms what is called the *tangent cone* through the given point.

N.B.—In general a surface generated by right lines which all pass through the same point is called a *cone*, and the point through which the lines pass is called its *vertex*. A cylinder (see p. 17) is the limiting case of a cone when the vertex is infinitely distant.

Ex. Show how to find the coordinates of the pole of a given plane with regard to a quadric $U = 0$.

64. The polar plane may be also defined as the locus of harmonic means of radii passing through the pole. In fact let us examine the locus of points of harmonic section of radii passing through the origin; then if ρ', ρ'' be the roots of the quadratic of Art. 60, and ρ the radius vector of the locus, we are to have

$$\frac{2}{\rho} = \frac{1}{\rho'} + \frac{1}{\rho''} = -\frac{2(\lambda l + \mu m + \nu n)}{d},$$

or, returning to x, y, z coordinates,

$$lx + my + nz + d = 0;$$

but this is the polar plane of the origin, as may be seen by making x', y', z' all $= 0$ in the equation written in full (Art. 62).

From this definition of the polar plane, it is evident that if a section of a surface be made by a plane passing through any point, the polar of that point with regard to the section will be the intersection of the plane of section with the polar plane of the given point. For the locus of harmonic means of *all* radii passing through the point must include the locus of harmonic means of the radii which lie in the plane of section.

65. If the polar plane of any point A pass through B, then the polar plane of B will pass through A.

For since the equation of the polar plane is symmetrical with respect to xyz, $x'y'z'$, we get the same result whether we

substitute the coordinates of the second point in the equation of the polar plane of the first, or *vice versâ.*

The intersection of the polar planes of A and of B will be a line which we shall call the *polar line*, with respect to the surface, of the line AB. It is easy to see that the polar line of the line AB is the locus of the poles of all planes which can be drawn through the line AB. [It is the intersection of the two tangent planes at the points where AB meets the quadric.]

66. If in the original equation we had not only $d=0$, but also l, m, n each $=0$, then the equation of the tangent plane at the origin, found (Art. 61), becomes illusory, since every term vanishes; and no single plane can be called the tangent plane at the origin. In fact, the coefficient of ρ (Art. 60) vanishes whatever be the direction of ρ; therefore *every* line drawn through the origin meets the surface in two consecutive points, and the origin is said to be a double point on the surface.

In the present case, the equation denotes a cone whose vertex is the origin. In fact *every homogeneous equation in x, y, z represents a cone.* For if such an equation be satisfied by any coordinates x', y', z', it will be satisfied by the coordinates kx', ky', kz' (where k is any constant), that is to say, by the coordinates of every point on the line joining $x'y'z'$ to the origin. This line then lies wholly in the surface, which must therefore consist of a series of right lines drawn through the origin.

The equation of the tangent plane at any point of the cone now under consideration may be written in either of the forms

$$xU_1'+yU_2'+zU_3'=0,\ x'U_1+y'U_2+z'U_3=0.$$

The former (wanting an absolute term) shows that the tangent plane at every point on the cone passes through the origin; the latter form shows that the tangent plane at any point $x'y'z'$ touches the surface at every point of the line joining $x'y'z'$ to the vertex; for the equation will represent the same plane if we substitute kx', ky', kz' for x', y', z'.

When the point $x'y'z$ is not on the surface, the equation we have been last discussing represents the polar of that point, and it appears in like manner that the polar plane of every point [other than the vertex] passes through the vertex of the cone, and also that all points which lie on the same line passing through the vertex of a cone have the same polar plane.

[It is evident geometrically that the vertex is the pole of any plane not passing through the vertex.]

To find the polar plane of any point with regard to a cone we need only take any section through that point, and take the polar line of the point with regard to that section; then the plane joining this polar line to the vertex will be the polar plane required. For it was proved (Art. 64) that the polar plane must contain the polar line, and it is now proved that the polar plane must contain the vertex.

67. We can easily find the *condition that the general equation of the second degree should represent a cone.* For if it does it will be possible by transformation of coordinates to make the new l, m, n, d vanish. The coordinates of the new origin must therefore (Art. 59) satisfy the conditions

$$U_1'=0,\ U_2'=0,\ U_3'=0,\ U'=0,$$

which last combined with the others is equivalent to $U_4'=0$. And if we eliminate x', y', z' from the four equations

$$\begin{aligned} ax'+\ hy'+gz'+\ l&=0,\\ hx'+\ by'+fz'+m&=0,\\ gx'+\ fy'+cz'+\ n&=0,\\ lx'+my'+nz'+\ d&=0,\end{aligned}$$

we obtain the required condition in the form of the determinant

$$\begin{vmatrix} a, & h, & g, & l\\ h, & b, & f, & m\\ g, & f, & c, & n\\ l, & m, & n, & d\end{vmatrix}=0,$$

which, written at full length, is

$$abcd+2afmn+2bgnl+2chlm+2dfgh-bcl^2-cam^2-abn^2$$
$$-adf^2-bdg^2-cdh^2+f^2l^2+g^2m^2+h^2n^2-2ghmn-2hfnl-2fglm=0.$$

We shall often write this equation $\Delta=0$, and (as in *Conics*, Art. 151) shall call Δ the *discriminant* of the given quadric.

It will be found convenient hereafter to use the abbreviations A, B, C, D, $2F$, $2G$, $2H$, $2L$, $2M$, $2N$, to denote the differential coefficients of Δ taken with respect to a, b, c, &c. Thus

$$\begin{aligned}
A &= bcd + 2fmn - bn^2 - cm^2 - df^2,\\
B &= cda + 2gnl - cl^2 - an^2 - dg^2,\\
C &= dab + 2hlm - am^2 - bl^2 - dh^2,\\
D &= abc + 2fgh - af^2 - bg^2 - ch^2,\\
F &= amn + dgh - adf + fl^2 - hnl - glm,\\
G &= bnl + dhf - bdg + gm^2 - flm - hmn,\\
H &= clm + dfg - cdh + hn^2 - gmn - fnl,\\
L &= bgn + chm - bcl + lf^2 - hfn - gfm,\\
M &= chl + afn - cam + mg^2 - fgl - ghn,\\
N &= afm + bgl - abn + nh^2 - ghm - hfl.
\end{aligned}$$

[These quantities are the first minors, with proper signs attached, of the original determinant Δ.]

[If $U=0$ be expressed in general quadriplanar or homogeneous co-ordinates (see Art. 62), the condition for a cone is still $\Delta=0$. In fact Δ is an *invariant* for all linear transformations. If we put

$$x = \lambda_1 X + \lambda_2 Y + \lambda_3 Z + \lambda_4 W,$$

the transformed value of Δ is the original Δ multiplied by the square of the determinant $(\lambda_1\mu_2\nu_3\rho_4)$. $\Delta=0$ is, analytically, the condition that U may be expressed as a homogeneous function of three variables by linear transformation. The corresponding determinant for a homogeneous quadric in n variables, when equated to zero, gives the condition that the quadric may be expressed homogeneously in $n-1$ variables.*]

68. Let us return now to the quadratic of Art. 60, in which d is not supposed to vanish, and let us examine the condition that the radius vector should be bisected at the origin. It is obviously necessary and sufficient that the coefficient of ρ in that quadratic should vanish, since we should then get for ρ values equal with opposite signs. The condition required then is

$$l\lambda + m\mu + n\nu = 0,$$

* See Bromwich, *Quadratic Forms* (Cambridge, 1906), p. 8.

which multiplied by ρ shows that the radius vector must lie in the plane $lx+my+nz=0$. Hence (Art. 64) *every right line drawn through a point in a plane parallel to its polar plane is bisected at the point.* [Hence this plane meets the quadric in a conic having the point for centre.]

69. If, however, we had $l=0$, $m=0$, $n=0$, then *every* line drawn through the origin would be bisected and the origin would be called the *centre* of the surface. *Every quadric has in general one and but one centre.* For if we seek by transformation of coordinates to make the new l, m, $n=0$, we obtain three equations, viz.

$$U'_1=0, \text{ or } ax'+hy'+gz'+\ l=0,$$
$$U'_2=0, \text{ or } hx'+by'+fz'+m=0,$$
$$U'_3=0, \text{ or } gx'+fy'+cz'+\ n=0,$$

which are sufficient to determine the three unknowns x', y', z'. The resulting values are $x'=\frac{L}{D}$, $y'=\frac{M}{D}$, $z'=\frac{N}{D}$, where L, M, N, D have the same meaning as in Art. 67.

If, however, $D=0$, the coordinates of the centre become infinite and the surface has no finite centre. If we write the original equation $u_2+u_1+u_0=0$, it is evident that D is the discriminant of u_2.*

* It is possible that the numerators of these fractions might vanish at the same time with the denominator, in which case the coordinates of the centre would become indeterminate, and the surface would have an infinity of centres. Thus if the three planes U_1, U_2, U_3 all pass through the same line, any point on this line will be a centre. The conditions that this should be the case may be written

$$\left\| \begin{matrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \end{matrix} \right\| = 0,$$

the notation indicating that all the four determinants must $=0$, which are got by erasing any of the vertical lines. [i.e. $L=0$, $M=0$, $N=0$, $D=0$. By using identities of the form $AD-L^2=(bc-f^2)\,\Delta$, it is seen that these four equations are equivalent to $D=0$, $\Delta=0$.] We shall reserve the fuller discussion of these cases for the next chapter.

70. *To find the locus of the middle points of chords parallel to a given line* $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$.

If we transform the equation to any point on the locus as origin, the new l, m, n must fulfil the condition (Art. 68) $l\lambda + m\mu + n\nu = 0$, and therefore (Art. 59) the equation of the locus is

$$\lambda U_1 + \mu U_2 + \nu U_3 = 0.$$

This denotes a plane through the intersection of the planes U_1, U_2, U_3, that is to say, through the centre of the surface. It is called the *diametral plane conjugate to the direction* of the chords.

If $x'y'z'$ be any point on the radius vector drawn through the origin parallel to the given direction, the equation of the diametral plane may be written

$$x'U_1 + y'U_2 + z'U_3 = 0.$$

If now we take the equation of the polar plane of kx', ky', kz',

$$kx'U_1 + ky'U_2 + kz'U_3 + U_4 = 0,$$

divide it by k, and then make k infinite, we see that the diametral plane is the polar of the point at infinity on a line drawn in the given direction, as we might also have inferred from geometrical considerations (see *Conics*, Art. 324).

In like manner, *the centre is the pole of the plane at infinity*, for if the origin be the centre, its polar plane (Art. 64) is $d = 0$, which (Art. 30) represents a plane situated at an infinite distance.

In the case where the given surface is a cone, it is evident that the plane which bisects chords parallel to any line drawn through the vertex is the same as the polar plane of any point in that line. In fact it was proved that all points on the line have the same polar plane, therefore the polar of the point at infinity on that line is the same as the polar plane of any other point in it.

71. The plane which bisects chords parallel to the axis of

x is found by making $\mu = 0$, $\nu = 0$ in the equation of Art. 70, to be

$$U_1 = 0, \text{ or } ax + hy + gz + l = 0,^*$$

and this will be parallel to the axis of y, if $h = 0$. But this is also the condition that the plane conjugate to the axis of y should be parallel to the axis of x. Hence *if the plane conjugate to a given direction be parallel to a second given line, the plane conjugate to the latter will be parallel to the former.*

When $h = 0$, the axes of x and y are evidently parallel to a pair of conjugate diameters of the section by the plane of xy: and it is otherwise evident that the plane conjugate to one of two conjugate diameters of a section passes through the other. For the locus of middle points of *all* chords of the surface parallel to a given line must include the locus of the middle points of all such chords which are contained in a given plane.

Three diametral planes are said to be conjugate when each is conjugate to the intersection of the other two, and three diameters are said to be conjugate when each is conjugate to the plane of the other two. Thus we should obtain a system of three conjugate diameters by taking two conjugate diameters of any central section together with the diameter conjugate to the plane of that section. If we had in the equation $f = 0$, $g = 0$, $h = 0$, it appears from the commencement of this article that the coordinate planes are parallel to three conjugate diametral planes.

When the surface is a cone, it is evident from what was said (Arts. 66, 70) that a system of three conjugate diameters meets any plane section in points such that each is the pole with respect to the section of the line joining the other two, i.e. the points form a self-conjugate triangle with regard to the section.

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* It follows that the plane $x = 0$ will bisect chords parallel to the axis of x, if $h = 0$, $g = 0$, $l = 0$; or, in other words, if the original equation do not contain any odd power of x. But it is otherwise evident that this must be the case in order that for any assigned values of y and z we may obtain equal and opposite values of x.

72. A diametral plane is said to be *principal* if it be perpendicular to the chords to which it is conjugate.

The axes being rectangular, and λ, μ, ν the direction-cosines of a chord, we have seen (Art. 70) that the corresponding diametral plane is

$$\lambda(ax+hy+gz+l)+\mu(hx+by+fz+m)+\nu(gz+fy+cz+n)=0,$$

and this will be perpendicular to the chord, if (Art. 45) the coefficients of x, y, z be respectively proportional to λ, μ, ν. This gives us the three equations

$$\lambda a+\mu h+\nu g=k\lambda,\quad \lambda h+\mu b+\nu f=k\mu,\quad \lambda g+\mu f+\nu c=k\nu.$$

From these equations, which are linear in λ, μ, ν, we can eliminate λ, μ, ν, when we obtain the determinant

$$\begin{vmatrix} a-k, & h, & g \\ h, & b-k, & f \\ g, & f, & c-k \end{vmatrix}=0,$$

which expanded gives a cubic for the determination of k, viz.

$$k^3-k^2(a+b+c)+k(bc+ca+ab-f^2-g^2-h^2)$$
$$-(abc+2fgh-af^2-bg^2-ch^2)=0,$$

and the three values hence found for k being successively substituted in the preceding equations, we thence determine the corresponding values of λ, μ, ν. Hence, *a quadric has in general three principal diametral planes*, the three diameters perpendicular to which are called the *axes* of the surface. We shall discuss this equation more fully in the next chapter.

Ex. To find the principal planes of

$$7x^2+6y^2+5z^2-4xy-4yz=6.$$

The cubic for k is

$$k^3-18k^2+99k-162=0,$$

whose roots are 3, 6, 9. Now our three equations are

$$7\lambda-2\mu=k\lambda,\quad -2\lambda+6\mu-2\nu=k\mu,\quad -2\mu+5\nu=k\nu.$$

If in these we substitute $k=3$, we find $2\lambda=\mu=\nu$. Multiplying by ρ, and substituting x for $\lambda\rho$, &c., we get for the equations of one of the axes $2x=y=z$. And the plane drawn through the origin (which is the centre), perpendicular to this line is $x+2y+2z=0$. In like manner the other two principal planes are $2x-2y+z=0$, $2x+y-2z=0$.

73. *The sections of a quadric by parallel planes are similar to each other.*

Since any plane may be taken for the plane of xy, it is

sufficient to consider the section made by it, which is found by putting $z=0$ in the equation of the surface. But the section by any parallel plane is found by transforming the equation to parallel axes through any new origin, and then making $z=0$.

If we retain the planes yz and zx, and transfer the plane xy parallel to itself, the section by this plane is got at once by writing $z=c$ in the equation of the surface, since it is evident that it is the same thing whether we write $z+c$ for z, and then make $z=0$, or whether we write at once $z=c$.

And since the coefficients of x^2, xy, and y^2 are unaltered by this transformation, the curves are similar.

It is easy to prove algebraically, that the locus of centres of parallel sections is the diameter conjugate to their plane, as is geometrically evident. [Obviously the tangent plane at the extremity of the diameter is parallel to these sections.]

74. If ρ', ρ'' be the roots of the quadratic of Art. 60, their product $\rho'\rho''$ is $=d$ divided by the coefficient of ρ^2. But if we transform to parallel axes, and consider a radius vector drawn parallel to the first direction, the coefficient of ρ^2 remains unchanged, and the product is proportional to the new d. Hence, if through two given points A, B, any parallel chords be drawn meeting the surface in points R, R'; S, S', then the products $RA \cdot AR'$, $SB \cdot BS'$ are to each other in a constant ratio, namely, $U' : U''$ where U', U'' are the results of substituting the coordinates of A and of B in the given equation.

75. We shall now show that the theorems already deduced from the discussion of lines passing through the origin may be derived by a more general process (such as that employed, *Conics*, Art. 91). For symmetry we use homogeneous equations with four variables. [The Cartesian forms follow by putting $w=1$ after the general forms have been deduced.]

To find the points where a given quadric is met by the line joining two given points $x'y'z'w'$, $x''y''z''w''$.

Let us take as our unknown quantity the ratio $\mu : \lambda$, in which the joining line is cut at the point where it meets the quadric, then (Art. 8) the coordinates of that point are proportional to

$$\lambda x' + \mu x'',\ \lambda y' + \mu y'',\ \lambda z' + \mu z'',\ \lambda w' + \mu w'';$$

and if we substitute these values in the equation of the surface, we get for the determination of $\lambda : \mu$, *a quadratic*

$$\lambda^2 U' + 2\lambda\mu P + \mu^2 U'' = 0.$$

The coefficients of λ^2 and μ^2 are easily seen to be the results of substituting in the equation of the surface the coordinates of each of the points, while the coefficients of $2\lambda\mu$ may be seen (by Taylor's theorem, or otherwise) to be capable of being written in either of the forms

$$x'U_1'' + y'U_2'' + z'U_3'' + w'U_4'',$$
$$x''U_1' + y''U_2' + z''U_3' + w''U_4'.$$

Having found from this quadratic the values of $\lambda : \mu$, substituting each of them in the expressions $\lambda x' + \mu x''$, &c., we find the coordinates of the points where the quadric is met by the given line.

76. If $x'y'z'w'$ be on the surface, then $U' = 0$, and one of the roots of the last quadratic is $\mu = 0$, which corresponds to the point $x'y'z'w'$, as evidently ought to be the case. In order that the second root should also be $\mu = 0$, we must have $P = 0$. If then the line joining $x'y'z'w'$ to $x''y''z''w''$ touch the surface at the former point, the coordinates of the latter must satisfy the equation

$$xU_1' + yU_2' + zU_3' + wU_4' = 0,$$

and since $x''y''z''w''$ may be *any* point on *any* tangent line through $x'y'z'w'$, it follows that every such tangent lies in the plane whose equation has been just written.

77. If $x'y'z'w'$ be *not* on the surface, and yet the relation $P = 0$ be satisfied, the quadratic of Art. 75 takes the form $\lambda^2 U' + \mu^2 U'' = 0$, which gives values of $\lambda : \mu$, equal with opposite signs. Hence the line joining the given points is cut by the surface externally and internally in the same ratio;

that is to say, is cut harmonically. It follows then that the locus of points of harmonic section of radii drawn through $x'y'z'w'$ is the polar plane

$$xU_1'+yU_2'+zU_3'+wU_4'=0.$$

78. In general if the line joining the two points touch the surface, the quadratic of Art. 75 must have equal roots, and the coordinates of the two points must be connected by the relation $U'U''=P^2$. If the point $x'y'z'w'$ be fixed, this relation ought to be fulfilled if the other point lie on any of the tangent lines which can be drawn through it. Hence the cone generated by all these tangent lines will have for its equation $UU'=P^2$, where

$$P=xU_1'+yU_2'+zU_3'+wU_4'.$$

Ex. 1. To find the equation of the tangent cone from the point $x'y'z'$ to the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Ans. $\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2.$

[Ex. 2. If the line joining $xyzw$ to $x'y'z'w'$ is cut harmonically by the two quadrics U and V then $UV'+U'V-2PQ=0$, where $Q=xV_1'+yV_2'+zV_3'+wV_4'$. Hence if $x'y'z'w'$ is fixed the locus of $xyzw$ is a cone with vertex at former point.

Ex. 3. If the line passes through the intersection of the quadrics U and V, then $4(UU'-P^2)(VV'-Q^2)-(UV'+U'V-2PQ)^2=0$, which represents a cone of the fourth degree with vertex at $x'y'z'w'$, and passing through the curve UV.

Ex. 4. The condition that U may be a cone with its vertex at $x'y'z'w'$ may be found by expressing the condition that the line joining $x'y'z'w'$ to any point meets the surface in two coincident points at the vertex. We get

$$U_1'=U_2'=U_3'=U_4'=0;\text{ hence }\Delta=0.]$$

79. *To find the condition that the plane $\alpha x+\beta y+\gamma z+\delta w=0$ should touch the surface given by the general equation.*

First, if x, y, z, w be the coordinates of the pole of this plane, and k an indeterminate multiplier, we have (Art. 63) in general

$$k\alpha=ax+hy+gz+lw,\quad k\beta=hx+by+fz+mw,$$
$$k\gamma=gx+fy+cz+nw,\quad k\delta=lx+my+nz+dw,$$

to determine the pole of the given plane. Solving for x, y, z, w from these equations, we find

$$\begin{aligned} \Delta x &= k(A\alpha + H\beta + G\gamma + L\delta), \\ \Delta y &= k(H\alpha + B\beta + F\gamma + M\delta), \\ \Delta z &= k(G\alpha + F\beta + C\gamma + N\delta), \\ \Delta w &= k(L\alpha + M\beta + N\gamma + D\delta), \end{aligned}$$

where Δ, A, B, C, &c., have the same meaning as in Art. 67. Now if these values satisfy the equation $\alpha x + \beta y + \gamma z + \delta w = 0$, we get by eliminating them

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2$$
$$+ 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta + 2L\alpha\delta + 2M\beta\delta + 2N\gamma\delta = 0,$$

which is the required condition that this plane should touch the surface.

The result of eliminating k, x, y, z, w from the four equations first written and $\alpha x + \beta y + \gamma z + \delta w = 0$ may evidently be written in the determinant form

$$\Sigma \equiv \begin{vmatrix} & \alpha, & \beta, & \gamma, & \delta \\ \alpha, & a, & h, & g, & l \\ \beta, & h, & b, & f, & m \\ \gamma, & g, & f, & c, & n \\ \delta, & l, & m, & n, & d \end{vmatrix} = 0.$$

Each of these is a form in which we may write the condition which must be satisfied by the coordinates of a plane if the plane touch the surface (see Art. 38); that is to say, *the tangential equation of the surface*, or the equation of the surface as an envelope of planes.

[If $U = 0$ represents a pair of planes, each of the ten first minors A, B, C, &c., vanishes. This may be proved analytically, and follows geometrically from the fact that, in this case, *every* plane touches $U = 0$ at the point where it meets the line of intersection of the two planes forming U. When the surface U represents two coincident planes all the second minors vanish (see p. 69 note).]

80. *To find the condition that the surface should be touched by any line*

$$\alpha x + \beta y + \gamma z + \delta w = 0, \ \alpha' x + \beta' y + \gamma' z + \delta' w = 0.$$

If the line touches, the equation of the tangent plane at the point of contact will be of the form

$$(a+\lambda a')x+(\beta+\lambda\beta')y+\&c.=0.$$

If then we write in the first four equations of the last article $a+\lambda a'$ for a, &c., and then between these equations and the two equations of the line, eliminate k, $k\lambda$, x, y, z, w, we have the result in the determinant form

$$\begin{vmatrix} & & a, & \beta, & \gamma, & \delta \\ & & a', & \beta', & \gamma', & \delta' \\ a, & a', & a, & h, & g, & l \\ \beta, & \beta', & h, & b, & f, & m \\ \gamma, & \gamma', & g, & f, & c, & n \\ \delta, & \delta', & l, & m, & n, & d \end{vmatrix} = 0.$$

This equation is of the second degree in the coefficients a, b, &c., and also in the six coordinates of the line.

If we calculate the leading minor of order 2 in the determinant reciprocal to that just written (*Higher Algebra*, Art. 33) we find that the condition is equivalent to

$$\Delta^{1}(\Sigma\Sigma'-\Pi^2)=0,$$

where $$2\Pi=a\frac{d\Sigma}{da}+\beta\frac{d\Sigma}{d\beta}+\gamma\frac{d\Sigma}{d\gamma}+\delta\frac{d\Sigma}{d\delta}.$$

We may see the equivalence of the two forms of this condition when Δ is not zero by remarking that $\Sigma\Sigma'-\Pi^2=0$ is the condition that the two tangent planes to the quadric which pass through the given line should coincide.

[By a similar method we can express as a determinant the *condition that the point of intersection of three planes L, M, N may lie on the quadric.* For the tangent plane at the point of intersection (x', y', z', w') must be of the form $\lambda L+\mu M+\nu N=0$. This gives four linear equations in $\lambda, \mu, \nu, x', y', z', w'$. Also $L'=0$, $M'=0$, $N'=0$. Hence eliminating these seven quantities we get a determinant equated to zero.]

[80*a*. *To find the condition that the line joining $xyzw$ to $x'y'z'w'$ may lie wholly on the surface.*

Evidently each of the coefficients in the quadratic of Art. 75 must vanish. Hence $U=0$, $P=0$, $U'=0$. But $P=0$ is

the tangent plane at $x'y'z'w'$, if x, y, z, w are variable. Hence *the tangent plane at any point on the surface meets the surface in two right lines.* For we have seen that it meets it in one right line, and its total intersection is a plane curve of the second order. The two lines intersect at the point of contact, since every line through this point lying in the tangent plane meets the surface in two coincident points.

A line lying wholly in a quadric is called a *rectilineal generator* or, briefly, a generator. *Every plane through a generator is a tangent plane at some point on the line;* for it meets the quadric in another line intersecting the former, and is therefore a tangent plane at the point of intersection.

80*b*. *To find the condition that the line of intersection of two planes $L=0$, $M=0$, may be wholly on the surface.*

From the preceding it follows that this is equivalent to the condition that every plane through $L=0$, $M=0$, may be a tangent plane. Hence $\lambda L+\mu M$ is a tangent plane for all values of $\lambda:\mu$. Substituting $\lambda a+\mu a'$ in the tangential equation $\Sigma=0$ (Art. 79) we get $\lambda^2\Sigma+2\lambda\mu\Pi+\mu^2\Sigma'=0$. Therefore the condition is $\Sigma=0$, $\Pi=0$, $\Sigma'=0$.

We see that the condition (Art. 80) that the line L, M, may touch the quadric is equivalent to the condition that the above quadratic may have two equal roots.

$\Pi=0$ is in general the tangential equation of the pole of the plane $M=0$, whose plane-coordinates are a', β', γ', δ'. If $\Sigma'=0$, then $\Pi=0$ represents the point of contact of the tangent plane M.]

Line Coordinates and Quadrics.

80*c**. *Given the six coordinates of any right line* (p, q, r, s, t, u) *to determine the coordinates of its polar line* (Art. 65).

Since the polar line is the intersection of the polar planes of the two points determining the ray (Art. 53*a*),

$$U_1'x+U_2'y+U_3'z+U_4'w=0,$$
$$U_1''x+U_2''y+U_3''z+U_4''w=0,$$

* The rest of this chapter may be omitted on first reading.

its coordinates as an axis (Art. 53*b*) are

$$\pi' = (U_2'U_3''),\ \kappa' = (U_3'U_1''),\ \rho' = (U_1'U_2''),$$
$$\sigma' = (U_1'U_4''),\ \tau' = (U_2'U_4''),\ \upsilon' = (U_3'U_4'').$$

Now if we expand

$$(U_2'U_3'') = \left\| \begin{matrix} h, & b, & f, & m \\ g, & f, & c, & n \end{matrix} \right\| \cdot \left\| \begin{matrix} x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \end{matrix} \right\|$$

as in Art. 53*e*, and the others likewise, we get, by a transformation of line coordinates, from the ray coordinates of one line the axial coordinates of its polar line. All the coefficients are the second minors of a determinant of the fourth order—in this case a symmetrical one, viz. the discriminant of the quadric.

As it is sometimes convenient to have abbreviations to denote these second minors of the discriminant in the determinant form of Art. 67, we shall adopt a double suffix notation ($a_{mn} = a_{nm}$), thus writing the axial coordinates or their corresponding ray coordinates in the form

$$\pi' = a_{11}p + a_{12}q + a_{13}r + a_{14}s + a_{15}t + a_{16}u = s',$$
$$\kappa' = a_{21}p + a_{22}q + a_{23}r + a_{24}s + a_{25}t + a_{26}u = t',$$
$$\rho' = a_{31}p + a_{32}q + a_{33}r + a_{34}s + a_{35}t + a_{36}u = u',$$
$$\sigma' = a_{41}p + a_{42}q + a_{43}r + a_{44}s + a_{45}t + a_{46}u = p',$$
$$\tau' = a_{51}p + a_{52}q + a_{53}r + a_{54}s + a_{55}t + a_{56}u = q',$$
$$\upsilon' = a_{61}p + a_{62}q + a_{63}r + a_{64}s + a_{65}t + a_{66}u = r'.*$$

Now, if we multiply these equations in order by p, q, r, s, t, u and add, the quantity on the right side vanishes if the line intersect its polar line (53*c*); but this happens only when the given line is a tangent to one of the plane sections through itself, that is, when it touches the surface. In this case, therefore, each of the lines touches the surface in their common point.

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* The following are the values of the coefficients a_{11}, a_{12}, &c., as they stand in the above equations:—

$$\begin{matrix} bc - f^2, & fg - ch, & hf - bg, & hn - gm, & bn - fm, & fn - cm, \\ fg - ch, & ca - g^2, & gh - af, & gl - an, & fl - hn, & cl - gn, \\ hf - bg, & gh - af, & ab - h^2, & am - hl, & hm - bl, & gm - fl, \\ hn - gm, & gl - an, & am - hl, & ad - l^2, & hd - ml, & gd - nl, \\ bn - fm, & fl - hn, & hm - bl, & hd - ml, & bd - m^2, & fd - nm, \\ fn - cm, & cl - gn, & gm - fl, & gd - nl, & fd - nm, & cd - n^2. \end{matrix}$$

Thus the *condition that the right line should touch is* $a_{11}p^2 + \&c. + a_{66}u^2 + 2a_{12}pq + \ldots + 2a_{56}tu = 0$, or briefly $\Psi = 0$. This can also be derived from the condition in Art. 78, which may be written

$$\left\| \begin{matrix} U_1', & U_2', & U_3', & U_4' \\ U_1'', & U_2'', & U_3'', & U_4'' \end{matrix} \right\| \cdot \left\| \begin{matrix} x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \end{matrix} \right\| = 0,$$

and reduced by the process of this article, the quantity on the left is found to be Ψ.

80*d*. The same problem may be treated as follows if the right line be given as the intersection of two planes

$$\alpha x + \beta y + \gamma z + \delta w = 0, \ \alpha' x + \beta' y + \gamma' z + \delta' w = 0.$$

Forming the coordinates of the right line joining their poles (Art. 79) we have, for instance, omitting a common multiplier,

$$p' = \left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| \cdot \left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix} \right\|,$$

which we may write

$$p' = a_{11}\pi + a_{12}\kappa + a_{13}\rho + a_{14}\sigma + a_{15}\tau + a_{16}\upsilon = \sigma',$$
$$q' = \&c. \qquad\qquad = \tau', \ \&c.,$$

where $BC - F^2 = a_{11}$, &c. But (*Higher Algebra*, Art. 33, or Burnside and Panton, Art. 146) this $= \Delta\ (ad - l^2) = \Delta a_{44}$, and so for each of the others. We thus see how to solve the six equations in the last article. To find p, for instance, we must multiply in order by a_{41}, a_{54}, a_{64}, a_{14}, a_{24}, a_{34}, and add; this gets

$$a_{14}\sigma' + a_{24}\tau' + a_{34}\upsilon' + a_{44}\pi' + a_{54}\kappa' + a_{64}\rho' = \Delta p$$
$$= a_{14}p' + a_{24}q' + a_{34}r' + a_{44}s' + a_{54}t' + a_{64}u'.$$

As before, this right line (axis) meets the polar right line (axis) when each touches the surface; thus the condition that this may happen may be written in any of the forms

$$a_{11}\pi^2 + \ldots + a_{66}\upsilon^2 + 2a_{12}\pi\kappa + \ldots + 2a_{56}\tau\upsilon = 0,$$
$$a_{11}p'^2 + a_{44}s'^2 + \ldots + 2a_{12}p'q' + \ldots + \ldots = 0,$$
$$\text{or } a_{44}\pi'^2 + a_{11}\sigma'^2 + \ldots + 2a_{12}\sigma'\tau' + \ldots + \ldots = 0.$$

80*e*. *To determine the points of contact of tangent planes through the line* (p, q, r, s, t, u) *to the quadric.*

The coordinates of the plane determined by three points $xyzw$, $x_1y_1z_1w_1$, $x_2y_2z_2w_2$ are found by solving between the equations

$$\begin{aligned} Ax + By + Cz + Dw &= 0, \\ Ax_1 + By_1 + Cz_1 + Dw_1 &= 0, \\ Ax_2 + By_2 + Cz_2 + Dw_2 &= 0, \end{aligned}$$

and with θ an undetermined multiplier we may write them, introducing the coordinates p, q, r, s, t, u of the line 1, 2

$$\begin{aligned} yu - zt + wp &= \theta A, \\ -xu \quad + zs + wq &= \theta B, \\ xt - ys \quad + wr &= \theta C, \\ xp + yq + zr \quad &= -\theta D. \end{aligned}$$

These may be regarded as equations determining the coordinates of any plane passing through the right line by means of the coordinates of any definite point not upon the right line, through which the plane is to pass.

Now if in the equations just written we assume that $A : B : C : D$ are the values of $U_1 : U_2 : U_3 : U_4$ for the point, this amounts to inquiring what is the point whose polar plane passes through the point itself and through the given right line; in other words, the point of contact of a tangent plane through the given line.

Thus, by eliminating x, y, z, w we get, to determine θ, the biquadratic

$$\begin{vmatrix} \theta a, & \theta h - u, & \theta g + t, & \theta l - p \\ \theta h + u, & \theta b, & \theta f - s, & \theta m - q \\ \theta g - t, & \theta f + s, & \theta c, & \theta n - r \\ \theta l + p, & \theta m + q, & \theta n + r, & \theta d \end{vmatrix} = 0,$$

which evidently reduces to a pure quadratic, and this is found to be $\theta^2 \Delta + \Psi = 0$. Substituting θ from this equation, we determine the coordinates x, y, z, w of the point of contact by solving between any three of the four following equations

$$\begin{aligned} \theta a \,.\, x + (\theta h - u)y + (\theta g + t)z + (\theta l - p)w &= 0, \\ (\theta h + u)x + \&c. \qquad &= 0, \ \&c. \end{aligned}$$

The two points of contact arise from the double sign

$$\theta \sqrt{(\Delta)} = \pm \sqrt{(-\Psi)}.$$

Now if we solve the quadratic of Art. 75 we find under the radical the quantity, $-\Psi$, as noticed in Art. 80c. Hence we may draw the following inferences as to the reality of the intersections of a right line with a quadric, and of the tangent planes which may be drawn through it, viz. we have, taking

Δ positive, Ψ positive; intersections imaginary, contacts imaginary:

for Δ positive, Ψ negative; intersections real, contacts real:

for Δ negative, Ψ positive; intersections imaginary, contacts real:

for Δ negative, Ψ negative; intersections real, contacts imaginary.

As the contacts coincide if $\Psi = 0$ this establishes once more the relation that the line may touch.

80*f*. We have thus found that whether considered as a ray or as an axis the coordinates of any line touching a surface of the second degree satisfy a relation of the second order. We saw already (Art. 53*c*) that in like manner the coordinates of any line which meets a given line satisfy a relation of the first order. But in neither case is the relation the most general one of its order which can subsist between those six coordinates. In fact, we saw that instead of the coordinates of the fixed right line being perfectly arbitrary, the universal relation of line coordinates must subsist between them. And again, the relation of the second degree just found instead of containing the full number (21) of independent constants, has that number of coefficients indeed, but all of them are functions of the 10 coefficients in the equation of the quadric surface touched.

Plücker has applied the term *complex of lines* to the entire system of lines which satisfy a single relation. In the case of the complex of lines which satisfy a homogeneous relation of the first degree between the six ray coordinates of a line, by supposing fixed one of the points determining any ray, we evidently get the equation of a plane through that point. If

we replace the ray coordinates by the axial coordinates, on supposing one of the planes determining the line fixed, we have the equation of a point in that plane. In like manner, for a relation of the second degree, the ray coordinates give, for a fixed point, a cone of the second degree with the fixed point as vertex, and, the axial coordinates, taking a fixed plane through the axis, give a conic section in that plane. In particular if the relation be that establishing contact between the right line and a quadric surface, the cone becomes the tangent cone from the special point, and the conic the conic of intersection of the special plane.

80*g*. *To find the conditions that a right line be wholly contained in the surface.*

It should be observed that whereas in plane geometry we cannot have in the quadratic of Art. 75 each of the coefficients zero without a certain relation holding between the coefficients of the conic, in quadric surfaces the vanishing of those coefficients implies no such relation. In fact, if we write down $U'=0$, $P=0$, $U''=0$ in full, as

$$\begin{aligned}
U_1'x' + U_2'y' + U_3'z' + U_4'w' &= 0,\\
U_1'x'' + U_2'y'' + U_3'z'' + U_4'w'' &= 0,\\
U_1''x' + U_2''y' + U_3''z' + U_4''w' &= 0,\\
U_1''x'' + U_2''y'' + U_3''z'' + U_4''w'' &= 0,
\end{aligned}$$

we see (as in Art. 53*b*) that they imply only the identity of the line joining the two points with its polar line. Thus as the quadratic in $\lambda:\mu$ is now indeterminate the line is wholly contained in the surface.

We noticed (Art. 80*e*) regarding the condition for contact that $\Psi = U'U'' - P^2$. Hence, differentiating Ψ in succession with regard to each of the coefficients of the quadric, as each result is of the form $\theta\, U' + \phi\, U'' + \chi\, P$, we see, that for a line to be wholly contained in the quadric, its coordinates satisfy each of the ten relations $\dfrac{d\Psi}{da}=0$, &c., $\dfrac{d\Psi}{df}=0$, &c., and these amount to no more than three independent relations.

[Ex. If (p, q, r, s, t, u) is a generator then the following six equations are satisfied

$$\frac{d\Psi}{dp} = ks, \quad \frac{d\Psi}{dq} = kt, \quad \frac{d\Psi}{dr} = ku$$

$$\frac{d\Psi}{ds} = kp, \quad \frac{d\Psi}{dt} = kq, \quad \frac{d\Psi}{du} = kr$$

where $k = \pm 2\sqrt{\Delta}$.

Prove that generators for which $k = +2\sqrt{\Delta}$ intersect all those for which $k = -2\sqrt{\Delta}$ (R. Russell).]

CHAPTER V.

CLASSIFICATION OF QUADRICS.

81. Our object in this chapter is the reduction of any equation of the second degree in three variables to the simplest form of which it is susceptible, and the classification of the different surfaces which it is capable of representing.

Let us commence by supposing the quantity which we called D (Art. 67) *not* to be $=0$. By transforming the equation to parallel axes through the centre, the coefficients l, m, n are made to vanish, and the equation becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0,$$

where d' is the result of substituting the coordinates of the centre in the equation of the surface. Remembering that

$$U' = x'U_1' + y'U_2' + z'U_3' + w'U_4',$$

and that the coordinates of the centre make U_1', U_2', U_3' vanish, it is easy to calculate that

$$d' = \frac{lL + mM + nN + dD}{D} = \frac{\Delta}{D},$$

where Δ, D, L, M, N have the same meaning as in Art. 67.

82. Having by transformation to parallel axes made the coefficients of x, y, z vanish, we can next make the coefficients of yz, zx, and xy vanish by changing the direction of the axes, retaining the new origin; and so reduce the equation to the form

$$a'x^2 + b'y^2 + c'z^2 + d' = 0.$$

It is easy to show from Art. 17 that we have constants enough at our disposal to effect this reduction, but the method we shall follow is the same as that adopted, *Conics*, Art. 157, namely, to prove that there are certain functions of the coefficients which remain unaltered when we transform from

one rectangular system to another, and by the help of these relations to obtain the actual values of the new a, b, c.

Let us suppose that by using the most general transformation which is of the form

$$x = \lambda\bar{x} + \mu\bar{y} + \nu\bar{z},\ y = \lambda'\bar{x} + \mu'\bar{y} + \nu'\bar{z},\ z = \lambda''\bar{x} + \mu''\bar{y} + \nu''\bar{z},$$

the function $\quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

becomes $\quad a'\bar{x}^2 + b'\bar{y}^2 + c'\bar{z}^2 + 2f'\bar{y}\bar{z} + 2g'\bar{z}\bar{x} + 2h'\bar{x}\bar{y},$

which we write for shortness $U = \bar{U}$. And if both systems of coordinates be rectangular, we must have

$$x^2 + y^2 + z^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

which we write for shortness $S = \bar{S}$. Then if k be any constant we must have $kS - U = k\bar{S} - \bar{U}$. Now if the first side be resolvable into factors, so must also the second. The discriminants of $kS - U$ and of $k\bar{S} - \bar{U}$ must therefore vanish for the same values of k. But the first discriminant is

$$k^3 - k^2\ (a + b + c) + k\ (bc + ca + ab - f^2 - g^2 - h^2)$$
$$- (abc + 2fgh - af^2 - bg^2 - ch^2).$$

Equating, then, the coefficients of the different powers of k to the corresponding coefficients in the second, we learn that if the equation be transformed from one set of rectangular axes to another, we must have

$$a + b + c = a' + b' + c',$$
$$bc + ca + ab - f^2 - g^2 - h^2 = b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2,$$
$$abc + 2fgh - af^2 - bg^2 - ch^2 = a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2.*$$

83. The above three equations at once enable us to transform the equation so that the new f, g, h shall vanish, since they determine the coefficients of the cubic equation whose roots are the new a, b, c. This cubic is then

$$\dagger a'^3 - (a + b + c)\ a'^2 + (bc + ca + ab - f^2 - g^2 - h^2)\ a'$$
$$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0,$$

* There is no difficulty in forming the corresponding equations for oblique coordinates. We should then substitute for S (see Art. 19),

$$x^2 + y^2 + z^2 + 2yz \cos\lambda + 2zx \cos\mu + 2xy \cos\nu,$$

and proceeding exactly as in the text, we should form a cubic in k, the coefficients of which would bear to each other ratios unaltered by transformation.

† This is the same cubic as that found, Art. 72, as the reader will easily see ought to be the case.

which may also be written

$$(a'-a)(a'-b)(a'-c)-f^2(a'-a)-g^2(a'-b)-h^2(a'-c) - 2fgh=0.$$

We give here Cauchy's proof that the roots of this equation are all real. The proof of a more general theorem, in which this is included, will be found in *Higher Algebra*, Art. 46 (or Burnside and Panton, Vol. II. p. 65).

Let the cubic be written in the form

$$(a'-a)\{(a'-b)(a'-c)-f^2\}-g^2(a'-b)-h^2(a'-c)-2fgh=0.$$

Let α, β be the values of a' which make $(a'-b)(a'-c)-f^2=0$, and it is easy to see that the greater of these roots α is greater than either b or c, and that the less root β is less than either.* Then if we substitute in the given cubic $a'=\alpha$, it reduces to

$$-\{(\alpha-b)g^2+2fgh+(\alpha-c)h^2\},$$

and since the quantity within the brackets is a perfect square in virtue of the relation $(\alpha-b)(\alpha-c)=f^2$, the result of substitution is essentially negative. But if we substitute $a'=\beta$, the result is

$$(b-\beta)g^2-2fgh+(c-\beta)h^2,$$

which is also a perfect square, and positive. Since then, if we substitute $a'=\infty$, $a'=\alpha$, $a'=\beta$, $a'=-\infty$, the results are alternately positive and negative, the equation has three real roots lying within the limits just assigned. The three roots are the coefficients of x^2, y^2, z^2 in the transformed equation, but it is of course arbitrary which shall be the coefficient of x^2 or of y^2, since we may call whichever axis we please the axis of x.

Central Surfaces.

84. Quadrics are classified according to the signs of the roots of the preceding cubic. [A quadric for which $D\neq 0$ is central, i.e. it has a unique centre at a finite distance from the origin.]

* We may see this either by actually solving the equation, or by substituting successively $a'=\infty$, $a'=b$, $a'=c$, $a'=-\infty$, when we get results $+$, $-$, $-$, $+$, showing that one root is greater than b, and the other less than c.

I. First, let all the roots be positive, and the equation can be transformed to

$$a'x^2 + b'y^2 + c'z^2 + d' = 0.*$$

The surface makes real intercepts on each of the three axes, and if the intercepts be a, b, c, it is easy to see that the equation of the surface may be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

As it is arbitrary which axis we take for the axis of x, we suppose the axis so taken that a the intercept on the axis of x may be the longest, and c the intercept on the axis of z may be the shortest.

The equation transformed to polar coordinates is

$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2},$$

which (remembering that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$) may be written in either of the forms

$$\frac{1}{\rho^2} = \frac{1}{a^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\beta + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\cos^2\gamma$$

$$= \frac{1}{c^2} - \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\cos^2\alpha - \left(\frac{1}{c^2} - \frac{1}{b^2}\right)\cos^2\beta,$$

from which it is easy to see that a is the maximum and c the minimum value of the radius vector. The surface is consequently limited in every direction, and is called an *ellipsoid*.

Every section of it is therefore necessarily also an ellipse. Thus the section by any plane $z = k$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$, and we shall obviously cease to have any real section when k is greater than c. The surface therefore lies altogether between the planes $z = \pm c$. Similarly for the other axes.

If two of the coefficients be equal (for instance, $a = b$), then all sections by planes parallel to the plane of xy are

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* I suppose in what follows that $d'\left(= \frac{\Delta}{D}, \text{Art. } 81\right)$ is negative. If it were positive we should only have to change all the signs in the equation. If it were $= 0$ the surface would represent a cone (Art. 67).

circles, and the surface is one of *revolution*, generated by the revolution of an ellipse round its axis major or axis minor, according as it is the two less or the two greater coefficients which are equal. These surfaces are also sometimes called the *prolate* and the *oblate* spheroid.

If all three coefficients be equal, the surface is a sphere.

85. II. Secondly, let one root of the cubic be negative. We may then write the equation in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

where a is supposed greater than b, and where the axis of z evidently does not meet the surface in real points. Using the polar equation

$$\frac{1}{\rho^2}=\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}-\frac{\cos^2\gamma}{c^2},$$

it is evident that the radius vector meets the surface or not according as the right-hand side of the equation is positive or negative; and that putting it $=0$ (which corresponds to $\rho=\infty$) we obtain a system of radii which separate the diameters meeting the surface from those that do not. We obtain thus the equation of the *asymptotic cone*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

[The asymptotic cone is to be regarded as the tangent cone from the centre to the surface. Hence, as we saw before (Art. 70), the centre is the pole of the plane at infinity. The asymptotic cone of the ellipsoid is imaginary, of this surface real.]

Sections of the surface parallel to the plane of xy are ellipses; those parallel to either of the other two principal planes are hyperbolas. [The section by the plane $y=b$ is a pair of real right lines $\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$, therefore $y=b$ is a tangent plane (Art. 80a). The sections by all planes parallel to $y=0$ are hyperbolas having their asymptotes parallel to these lines. If we project these sections orthogonally on the tangent

plane $y=b$, we get two conjugate systems of hyperbolas, corresponding to the sections by planes on opposite sides of the tangent plane; the asymptotes of these hyperbolas are the two right lines mentioned. Similarly for the sections by $x=a$.] The equation of the elliptic section by the plane $z=k$ being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$, we see that a real section is found whatever be the value of k, and therefore that the surface is continuous [i.e. it is possible to pass from one point to another on the surface without leaving the surface]. It is called the *Hyperboloid of one sheet.**

If $a=b$, it is a surface of revolution.

86. III. Thirdly, let two of the roots be negative, and the equation may be written

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The sections parallel to two principal planes are hyperbolas, while that parallel to the plane of yz is an ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1.$$

It is evident that this will not be real so long as k is within the limits $\pm a$, but that any plane $x=k$ will meet the surface in a real section provided k is outside these limits. No portion of the surface will then lie between the planes $x=\pm a$, but the surface will consist of two separate portions outside these boundary planes. This surface is called the *Hyperboloid of two sheets.* [It has a real asymptotic cone $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, but unlike the hyperboloid of two sheets, it has no real right lines lying on the surface.] It is of revolution if $b=c$.

By considering the surfaces of revolution, the reader can easily form an idea of the distinction between the two kinds of hyperboloids. Thus, if a common hyperbola revolve round

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* See illustration.

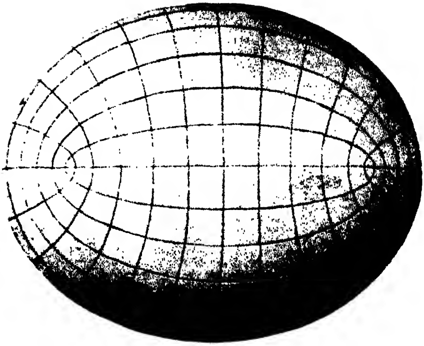

FIG. 1.

Ellipsoid with lines of curvature. (Arts. 84, 196, 302.)
The shortest axis is perpendicular to plane of paper.

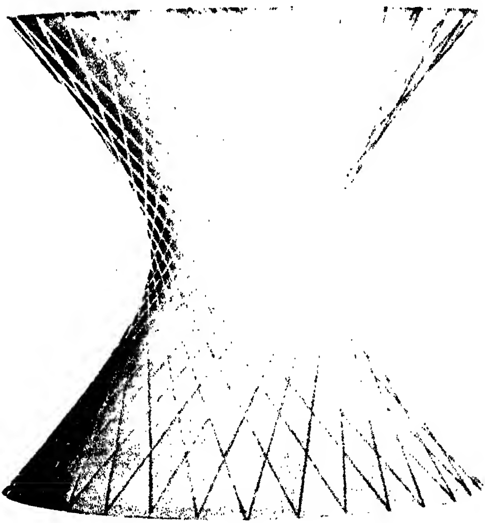

FIG. 2.

Hyperboloid of one sheet, with generators.
(Arts. 85, 107-109.)

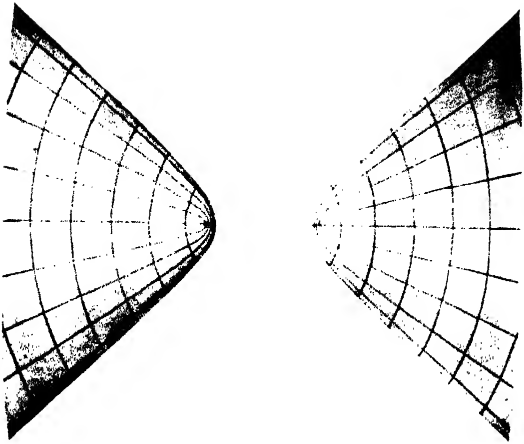

FIG. 3.
Hyperboloid of two sheets, with lines of curvature.
(Arts. 86, 196, 302.)

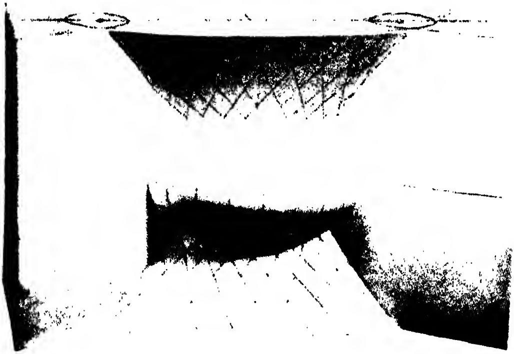

FIG. 4.
Three intersecting confocal quadrics. The lines on the ellipsoid are lines of curvature, those on the hyperboloid of one sheet are generators. (Arts. 159, 162, 196, 304.)

its transverse axis, the surface generated will evidently consist of two separate portions; but if it revolve round the conjugate axis it will consist but of one portion, and will be a case of the hyperboloid of one sheet.

IV. If the three roots of the cubic be negative, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$

can evidently be satisfied by no real values of the coordinates.

V. When the absolute term vanishes, we have the cone as a limiting case of the above. Forms I. and IV. then become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

which can be satisfied by no real values of the coordinates, while forms II. and III. give the equation of the cone in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

[The cone is the type separating the two kinds of hyperboloids and may be regarded as a limiting case of either.]

The forms already enumerated exhaust all the varieties of central surfaces.

[All sections of an ellipsoid are ellipses; the hyperboloids and cone have elliptic, parabolic, and hyperbolic sections, and it is natural to expect that the parabolic separate the other sections. In fact *all sections made by planes parallel to tangent planes to the asymptotic cone are parabolas*; those made by planes parallel to central planes meeting the asymptotic cone in distinct right lines are hyperbolas or ellipses according as these lines are real or imaginary. For all parallel planes meet the plane at infinity in the same line, and their sections therefore meet the plane at infinity in the same two points on the surface. A tangent plane to the asymptotic cone meets the plane at infinity in two coincident points on the surface; hence any parallel section is parabolic, because it touches the line at infinity in its own plane. The rest of the theorem follows from the fact that a hyperbola meets the line at infinity in its plane in two distinct real points, an ellipse in two distinct imaginary points. A cone of course coincides with its asymptotic cone. A pair of non-parallel right lines is a special case of a hyperbolic section; a pair of parallel right lines of a parabolic.]

Ex. 1. $7x^2 + 6y^2 + 5z^2 - 4yz - 4xy = 6.$

The discriminating cubic is $a'^3 - 18a'^2 + 99a' - 162 = 0$,
and the transformed equation $x^2 + 2y^2 + 3z^2 = 2$, an ellipsoid.

Ex. 2. $11x^2 + 10y^2 + 6z^2 - 12xy - 8yz + 4zx = 12.$

Discriminating cubic $a'^3 - 27a'^2 + 180a' - 324 = 0.$

Transformed equation $x^2 + 2y^2 + 6z^2 = 4$, an ellipsoid.

Ex. 3. $7x^2 - 13y^2 + 6z^2 + 24xy + 12yz - 12zx = \pm 84.$

Discriminating cubic $a'^3 - 343a' + 2058 = 0.$

Transformed equation $x^2 + 2y^2 - 3z^2 = \pm 12,$

a hyperboloid of one or of two sheets, according to the sign of the last term.

Ex. 4. $2x^2 + 3y^2 + 4z^2 + 6xy + 4yz + 8zx = 8.$

Discriminating cubic is $a'^3 - 9a'^2 - 3a' + 20 = 0.$

By Des Cartes's rule of signs this equation has two positive and one negative root, and therefore represents a hyperboloid of one sheet.

[Ex. 5. Prove that a tangent plane to the asymptotic cone meets a quadric in a pair of parallel straight lines.

Ex. 6. Prove that sections of a cone parallel to its tangent planes are parabolas.]

Non-central Surfaces.

87. Let us proceed now to the case where we have $D = 0$. In this case we have seen (Art. 69) that it is generally impossible by any change of origin to make the terms of the first degree in the equation to vanish. But it is in general quite indifferent whether we commence, as in Art. 69, by transforming to a new origin, and so remove the coefficients of x, y, z, or whether we first, as in this chapter, transform to new axes retaining the same origin, and so reduce the terms of highest degree to the form $a'x^2 + b'y^2 + c'z^2$. When $D = 0$, the first transformation being impossible, we must commence with the latter. And since the absolute term of the cubic of Art. 83 is D, one of its roots, that is to say, one of the three quantities a', b', c' must in this case $= 0$. The terms of the second degree are therefore reducible to the form $a'x^2 \pm b'y^2$. This is otherwise evident from the consideration that $D = 0$ is the condition that the terms of highest degree should be resolvable into two real or imaginary factors, in which case they may obviously be also expressed as the difference or sum of two squares. In this way the equation is reduced to the form

$$a'x^2 \pm b'y^2 + 2l'x + 2m'y + 2n'z + d = 0.$$

We can then, by transforming to a new origin, make the coefficients of x and y to vanish, but not that of z, and the

equation takes the form

$$a'x^2 \pm b'y^2 + 2n'z + d' = 0.$$

[The discriminating cubic is $p(p - a')\ (p \mp b') = 0$, and the discriminant Δ (Art. 67) is $\mp a'b'n'^2$. We consider first the cases for which $\Delta \neq 0$ and is either positive or negative.]

I. If $\Delta \neq 0$ then a', b', $n' \neq 0$. We can by a change of origin make the absolute term vanish, and reduce the equation to the form

$$a'x^2 \pm b'y^2 + 2n'z = 0.$$

Let us first suppose Δ negative and therefore the sign of b' positive assuming a' is always positive. In this case, while the sections by planes parallel to the planes of xz or yz are parabolas, those parallel to the plane of xy are ellipses, and the surface is called the *Elliptic Paraboloid.* It evidently extends only in one direction, since the section by any plane $z = k$ is $a'x^2 + b'y^2 = -2kn'$, and will not be real unless the right-hand side of the equation is positive. When therefore n' is positive, the surface lies altogether on the negative side of the plane of xy, and when n' is negative, on the positive side. If $a' = b'$ we get a paraboloid of revolution formed by rotating the parabola $a'x^2 + 2n'z$, $y = 0$ round the axis of z.

II. If Δ be positive and therefore b' negative (since a' is assumed positive) the sections by planes parallel to that of xy are hyperbolas, and the surface is called a *Hyperbolic Paraboloid.* This surface extends indefinitely in both directions. The section by the plane of xy is a pair of right lines; the parallel sections above and below this plane are hyperbolas having their transverse axes at right angles to each other, and their asymptotes parallel to the pair of lines in question, the section by the plane of xy forming the transition between the two series of hyperbolas: the form of the surface resembles a saddle. The sections by $x = 0$, $y = 0$ are parabolas.

III. [If $\Delta = 0$ we have either n', b' or $a' = 0$.] Suppose $n' = 0$. The equation then does not contain z, and therefore (Art. 25) represents a *cylinder which is elliptic or hyperbolic,* according as a' and b' have the same or different signs. Since

the terms of the first degree are absent from the equation the origin is a centre, but so is also equally every other point on the axis of z, which is called the axis of the cylinder. The possibility of the surface having a line of centres is indicated by both numerator and denominator vanishing in the coordinates of the centre, Art. 69, note.

If it happened that not only n' but also $d'=0$, the surface would reduce to *two intersecting planes*.

IV. If $b'=0$, that is if *two* roots of the discriminating cubic vanish, the equation takes the form

$$a'x^2+2m'y+2n'z+d=0,$$

but by changing the axes of y and z in their own plane, and taking for new coordinate planes the plane $m'y+n'z$ and a plane perpendicular to it through the axis of x, the equation is brought to the form

$$a'x^2+2m'y+d=0,$$

which (Art. 25) represents *a cylinder whose base is a parabola.*

V. If we have also $m'=0$, $n'=0$, the equation $a'x^2+d=0$ being resolvable into factors would evidently denote *a pair of parallel planes.*

[Thus if $D=0$, $\Delta \neq 0$, the general equation represents a paraboloid; if $D=0$, $\Delta=0$ it represents a cylinder, which may reduce to a pair of planes.]

[All sections of a paraboloid parallel to the axis (i.e. the axis of z when the paraboloid is written in the form $ax^2 \pm by^2 + 2nz = 0$) are parabolas. All other sections of the elliptic paraboloid are ellipses; and those of the hyperbolic paraboloid are hyperbolas.

To prove these theorems, use the general forms $L^2 \pm M^2 + P=0$, where L, M, P are planes, the line $L=0$, $M=0$, being parallel to the axis of the parabola, the positive sign corresponding to the elliptic and the negative to the hyperbolic type. Since the axes of coordinates are quite general, we get the type of a section by putting $z=0$, when we get an ellipse or hyperbola according as the sign is positive or negative; but in the special case when the plane z coincides with L or M (and is therefore parallel to the axis) the section becomes a parabola.]

88. The actual work of reducing the equation of a paraboloid to the form $a'x^2+b'y^2+2n'z=0$ is shortened by observing that the discriminant is an invariant; that is to say, a function of the coefficients which is not altered by transforma-

tion of coordinates (*Higher Algebra*, Art. 120, also noticing that since we are transforming from one set of rectangular axes to another the modulus of transformation is ± 1, as seen in Ex. 1, Art. 32). Now the discriminant of $a'x^2 + b'y^2 + 2n'z$ is simply $-a'b'n'^2$, which is therefore equal to the discriminant of the given equation. And as a' and b' are known, being the two roots of the discriminating cubic which do not vanish, n' is also known. The calculation of the discriminant is facilitated by observing that it is in this case a perfect square (*Higher Algebra*, Art. 37). Thus let us take the example

$$5x^2 - y^2 + z^2 + 6zx + 4xy + 2x + 4y + 6z = 8.$$

Then the discriminating cubic is $\lambda^3 - 5\lambda^2 - 14\lambda = 0$ whose roots are 0, 7, and -2. We have therefore $a' = 7$, $b' = -2$. The discriminant in this case is $(l + 2m - 3n)^2$, or putting in the actual values $l = 1$, $m = 2$, $n = 3$, is 16. Hence we have $14n'^2 = 16$, $n' = \frac{4}{\sqrt{(14)}}$, and the reduced equation is $7x^2 - 2y^2 = \frac{8z}{\sqrt{(14)}}$.

If we had not availed ourselves of the discriminant we should have proceeded, as in Art. 72, to find the principal planes answering to the roots 0, 7, -2 of the discriminating cubic, and should have found

$$x + 2y - 3z = 0,\ 4x + y + 2z = 0,\ x - 2y - z = 0.$$

Since the new coordinates are the perpendiculars on these planes, we are to take

$$4x + y + 2z = X\sqrt{(21)},\ x - 2y - z = Y\sqrt{(6)},\ x + 2y - 3z = Z\sqrt{(14)},$$

from which we can express x, y, z in terms of the new coordinates, and the transformed equation becomes

$$7x^2 - 2y^2 + \frac{24x}{\sqrt{(21)}} - 2\sqrt{(6)}\, y - \frac{8}{\sqrt{(14)}} z = 8,$$

which, finally transformed to parallel axes through a new origin, gives the same reduced equation as before.

If in the preceding example the coefficients l, m, n had been so taken as to fulfil the relation $l + 2m - 3n = 0$, the discriminant would then vanish, but the reduction could be effected with even greater facility, as the terms in x, y, z could then be expressed in the form

$$(4x + y + 2z) + \lambda(x - 2y - z).$$

Thus the equation

$$5x^2 - y^2 + z^2 + 6zx + 4xy + 2x + 2y + 2z = 8$$

may be written in the form

$$(4x + y + 2z)^2 - (x - 2y - z)^2 + 2(4x + y + 2z) - 2(x - 2y - z) = 24,$$

which, transformed as before, becomes

$$21x^2 - 6y^2 + 2x\sqrt{(21)} - 2y\sqrt{(6)} = 24,$$

and the remainder of the reduction presents no difficulty.

[Ex. A paraboloid we have seen has parabolic sections and either elliptic or hyperbolic sections. There are two corresponding ways of generating it.

(1) Take any conic (S), and draw a series of parabolas having for common axis the perpendicular to the plane of the conic through its centre which is to be the common vertex of the parabolas. Let the parameter of each parabola be proportional to the square of the central radius vector in which its plane meets the conic, then these parabolas generate a paraboloid, which is elliptic or hyperbolic according to the nature of the conic S.

(2) The paraboloid is also generated by a series of similar and similarly situated conics drawn parallel to a fixed plane, the centres of the conics lying in a line perpendicular to the plane, the squares of the axes of a conic being proportional to the distance of its centre from the fixed plane.

Analytical Classification of Quadrics.

88*a*. We can find independent and sufficient algebraic conditions that the general equation $U=0$ may represent a real quadric of any particular class, the case of two planes being to some extent an exception.

CENTRAL QUADRICS.

These are distinguished by the condition $D \neq 0$ (Art. 84) and the equation of a quadric for which this condition is satisfied is of the form

$$S + \frac{\Delta}{D} = 0$$

where the origin is at the centre (Art. 81) and S is the homogeneous function $(a, b, c, f, g, h)(x, y, z)^2$. We assume that $\frac{\Delta}{D}$ is negative. Now S may be expressed in an endless number of ways in the form $AX^2 + BY^2 + CZ^2$ where X, Y, Z are linear and homogeneous in x, y, z; but in whatever way it be thus expressed provided the transformation is real, the number of the coefficients A, B, C, having a given sign is always the same.* For if possible let the number be different for two forms of S, namely $AX^2 + BY^2 + CZ^2$ and $A'X'^2 + B'Y'^2 + C'Z'^2$, and suppose for example that A, B, C, A', B', are positive and that C' is negative. We have then $A'X'^2 + B'Y'^2 = AX^2 + BY^2 + CZ^2 - C'Z'^2$. We can find any number of real values of x, y, z such that X', Y'

* A similar theorem applies to a homogeneous equation in n variables (see Burnside and Panton's *Theory of Equations*, Art. 201).

will vanish, and for these values the sum of the four squares will be zero, which is impossible.

To discriminate therefore between the different central quadrics we have only to determine the number of positive coefficients occurring when we write S in the form

$$\frac{(ax + hy + gz)^2}{a} + \frac{\{y(ab - h^2) + z(af - gh)\}^2}{a(ab - h^2)} + \frac{z^2 D}{ab - h^2}.$$

Thus, assuming that a and $ab - h^2$ as well as D are different from zero, we have the following criteria: if a, $ab - h^2$ and D are all positive, the surface is an ellipsoid, reducing to an imaginary cone for $\Delta = 0$; if a and D are negative and $ab - h^2$ is positive, the surface is imaginary; in all the remaining cases the surface is a hyperboloid, reducing to a real cone for $\Delta = 0$, and the hyperboloid is of two sheets or of one according as D is positive or negative.

The argument fails if a or $ab - h^2$ is zero, but the criteria are still valid. That the surface cannot be an ellipsoid or an imaginary quadric follows from the fact that either of the conditions $a = 0$, $ab - h^2 = 0$ implies the existence of real points at infinity; and that the sign of D alone is still sufficient to discriminate between the remaining possibilities is easily proved by transforming to a new set of rectangular axes so that a and $ab - h^2$ are replaced by quantities different from zero while D and Δ are unaltered.

NON-CENTRAL QUADRICS.

The general criterion (Art. 87) is $D = 0$, since a root of the cubic (Art. 83) vanishes. Thus the criterion for a *paraboloid* is $D = 0$, $\Delta \neq 0$, the surface being elliptic or hyperbolic according as $bc + ca + ab - f^2 - g^2 - h^2$ is positive or negative; and the criterion for a *cylinder* is $D = 0$, $\Delta = 0$.

The cylinder will reduce to *a pair of planes* if all the ten first minors of Δ vanish (Art. 79). But since the equation of two planes contains 6 constants and that of the general equation contains 9, we should expect that *three* conditions were sufficient; in fact since Δ is zero, the first minors are connected by equations of the form, $AB - H^2 = 0$, $GM - FL = 0$, $lL + mM + nN + dD = 0$. Solving for x from the equation $U = 0$, and expressing the condition that the part under the radical may be a perfect square, we get $F = 0$, $M = 0$, $N = 0$. But in this process we have assumed $a \neq 0$, and $ab - h^2 \neq 0$, and if these do not hold, the three conditions are insufficient. The test is thus not universally applicable; for example all the first minors except D vanish for the cones $ax^2 + 2fyz = 0$, $fyz + gzx + hxy = 0$. The conditions are really alternative groups of conditions and we cannot state three conditions categorically, unless we are given $a \neq 0$, and $ab - h^2 \neq 0$ or two similar relations.

Ex. 1. Express the condition that the equation $fyz + gzx + hxy + lxw + myw + nzw = 0$ may represent a pair of planes, as an alternative between groups of equations. *Ans.* Either (i) $f = m = n = 0$ (or three similar equations) *or* (ii) $h = n = lf - gm = 0$ (or three similar equations). Here there are six alternatives, each expressing three relations.]

CHAPTER VI.

PROPERTIES OF QUADRICS DEDUCED FROM SPECIAL FORMS OF THEIR EQUATIONS.

Central Surfaces.

89. We proceed now to give some properties of central quadrics derived from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. This will include properties of the hyperboloids as well as of the ellipsoid if we suppose the signs of b^2 and of c^2 to be indeterminate.

The equation of the polar plane of the point $x'y'z'$ (or of the tangent plane, if that point be on the surface) is (Art. 63)

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

The length of the perpendicular from the origin on the tangent plane is therefore (Art. 33) given by the equation

$$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}.$$

And the angles α, β, γ which the perpendicular makes with the axes are given by the equations

$$\cos\alpha=\frac{px'}{a^2},\ \cos\beta=\frac{py'}{b^2},\ \cos\gamma=\frac{pz'}{c^2},$$

as is evident by multiplying the equation of the tangent plane by p, and comparing it with the form

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p.$$

From the preceding equations we can also immediately get an expression for the perpendicular in terms of the angles it makes with the axes, viz.

$$p^2=a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma.$$

90. *To find the condition that the plane $\alpha x + \beta y + \gamma z + \delta = 0$ should touch the surface.*

Comparing this with the equation $\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1$, we have at once

$$\frac{x'}{a} = -\frac{a\alpha}{\delta},\ \frac{y'}{b} = -\frac{b\beta}{\delta},\ \frac{z'}{c} = -\frac{c\gamma}{\delta},$$

and the required condition is

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 = \delta^2.$$

In the same way, the condition that the plane

$$\alpha x + \beta y + \gamma z = 0$$

should touch the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ is

$$a^2\alpha^2 + b^2\beta^2 - c^2\gamma^2 = 0.$$

These might also be deduced as particular cases of Art. 79.

91. The normal is a perpendicular to the tangent plane erected at the point of contact. Its equations are obviously

$$\frac{a^2}{x'}(x - x') = \frac{b^2}{y'}(y - y') = \frac{c^2}{z'}(z - z').$$

Let the common value of these be R, then we have

$$x - x' = \frac{Rx'}{a^2},\ y - y' = \frac{Ry'}{b^2},\ z - z' = \frac{Rz'}{c^2}.$$

Squaring, and adding, we find that the length of the normal between $x'y'z'$ and any point on it xyz is $\pm \frac{R}{p}$. But if xyz be taken as the point where the normal meets the plane of xy, we have $z = 0$, and the last of the three preceding equations gives $R = -c^2$. Hence the length of the intercept on the normal between the point of contact and the plane of xy is $\frac{c^2}{p}$.

92. *The sum of the squares of the reciprocals of any three rectangular diameters is constant.* This follows immediately from adding the equations

$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2},$$

$$\frac{1}{\rho'^2} = \frac{\cos^2\alpha'}{a^2} + \frac{\cos^2\beta'}{b^2} + \frac{\cos^2\gamma'}{c^2},$$

$$\frac{1}{\rho''^2} = \frac{\cos^2\alpha''}{a^2} + \frac{\cos^2\beta''}{b^2} + \frac{\cos^2\gamma''}{c^2},$$

whence, since $\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$, &c., we have

$$\frac{1}{\rho^2} + \frac{1}{\rho'^2} + \frac{1}{\rho''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

93. In like manner *the sum of the squares of three perpendiculars on tangent planes, mutually at right angles, is constant*, as appears from adding the equations

$$p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma,$$
$$p'^2 = a^2 \cos^2 \alpha' + b^2 \cos^2 \beta' + c^2 \cos^2 \gamma',$$
$$p''^2 = a^2 \cos^2 \alpha'' + b^2 \cos^2 \beta'' + c^2 \cos^2 \gamma''.$$

Hence *the locus of the intersection of three tangent planes which cut at right angles is a sphere (the director sphere)*; since the square of its distance from the centre of the surface is equal to the sum of the squares of the three perpendiculars, and therefore to $a^2 + b^2 + c^2$. [For the paraboloid the corresponding locus is a plane.]

Conjugate Diameters.

94. The equation of the diametral plane conjugate to the diameter drawn to the point $x'y'z'$ on the surface is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0, \text{ (Art. 70).}$$

It is therefore parallel to the tangent plane at that point. Since any diameter in the diametral plane is conjugate to that drawn to the point $x'y'z'$, it is manifest that when two diameters are conjugate to each other, their direction-cosines are connected by the relation

$$\frac{\cos\alpha\cos\alpha'}{a^2} + \frac{\cos\beta\cos\beta'}{b^2} + \frac{\cos\gamma\cos\gamma'}{c^2} = 0.$$

Since the equation of condition here given is not altered if we write ka^2, kb^2, kc^2 for a^2, b^2, c^2, it is evident that two lines which are conjugate diameters for any surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

are also conjugate diameters for any similar surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k.$$

And by making $k=0$ we see in particular that any surface and its asymptotic cone have common systems of conjugate diameters.

Following the analogy of methods employed in the case of conics, we may denote the coordinates of any point on the ellipsoid by $a\cos\lambda$, $b\cos\mu$, $c\cos\nu$, where λ, μ, ν are the direction-angles of some line; that is to say, are such that $\cos^2\lambda+\cos^2\mu+\cos^2\nu=1$. In this method the two lines answering to two conjugate diameters are at right angles to each other; for writing $\rho\cos\alpha=a\cos\lambda$, $\rho\cos\alpha'=a\cos\lambda'$, &c., the relation above written becomes

$$\cos\lambda\cos\lambda'+\cos\mu\cos\mu'+\cos\nu\cos\nu'=0.$$

95. *The sum of the squares of a system of three conjugate semi-diameters is constant.*

For the square of the length of any semi-diameter

$$x'^2+y'^2+z'^2$$

is, when expressed in terms of λ, μ, ν,

$$a^2\cos^2\lambda+b^2\cos^2\mu+c^2\cos^2\nu,$$

and the sum of this and

$$a^2\cos^2\lambda'+b^2\cos^2\mu'+c^2\cos^2\nu',$$

$$a^2\cos^2\lambda''+b^2\cos^2\mu''+c^2\cos^2\nu'',$$

is equal to $a^2+b^2+c^2$; since λ, μ, ν, &c., are the direction-angles of three lines mutually at right angles.

96. *The parallelepiped whose edges are three conjugate semi-diameters has a constant volume.*

For if $x'y'z'$, $x''y''z''$, &c. be the extremities of the diameters, the volume is (Art. 32)

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix},$$

or

$$abc\begin{vmatrix} \cos\lambda, & \cos\mu, & \cos\nu \\ \cos\lambda', & \cos\mu', & \cos\nu' \\ \cos\lambda'', & \cos\mu'', & \cos\nu'' \end{vmatrix},$$

but the value of the last determinant is ± 1 (see Ex. 1, Art. 82); hence the volume of the parallelepiped is abc.

If the axis of any central plane section be a', b', and p the perpendicular on the parallel tangent plane, then $a'b'p = abc$. For if c' be the semi-diameter to the point of contact, and θ the angle it makes with p, the volume of the parallelepiped under the conjugate diameters a', b', c' is $a'b'c' \cos\theta$, and $c' \cos\theta = p$.

97. The theorems just given may also with ease be deduced from the corresponding theorems for conics.

For consider any three conjugate diameters a', b', c' and let the plane of $a'b'$ meet the plane of xy in a diameter A, and let C be the diameter conjugate to A in the section $a'b'$, then we have $A^2 + C^2 = a'^2 + b'^2$; therefore $a'^2 + b'^2 + c'^2 = A^2 + C^2 + c'^2$. Again, since A is in the plane xy, then if B is the diameter conjugate to A in the section by that plane, the plane conjugate to A will be the plane containing B and containing the axis c, and C, c' are therefore conjugate diameters of the same section as B, c. Hence we have $A^2 + C^2 + c'^2 = A^2 + B^2 + c^2$; and since, finally, $A^2 + B^2 = a^2 + b^2$, the theorem is proved. Precisely similar reasoning proves the theorem about the parallelepipeds.

We might further prove these theorems by obtaining, as in the note, Art. 82, the relations which exist when the quantity $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2}$ in oblique coordinates is transformed to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ in rectangular coordinates. These relations are found to be

$$a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2,$$

$$b^2c^2 + c^2a^2 + a^2b^2 = b'^2c'^2 \sin^2\lambda + c'^2a'^2 \sin^2\mu + a'^2b'^2 \sin^2\nu,$$

$$a^2b^2c^2 = a'^2b'^2c'^2(1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu).$$

The first and last equations give the properties already obtained. The second expresses that the sum of the squares of the parallelograms formed by three conjugate diameters, taken two by two, is constant, or that the sum of squares of reciprocals of perpendiculars on tangent planes through three conjugate vertices is constant.

98. *The sum of the squares of the projections of three conjugate diameters on any fixed right line is constant.*

Let the line make angles α, β, γ with the axes, then the projection on it of the semi-diameter terminating in the point $x'y'z'$ is $x'\cos\alpha+y'\cos\beta+z'\cos\gamma$, or, by Art. 94, is

$$a\cos\lambda\cos\alpha+b\cos\mu\cos\beta+c\cos\nu\cos\gamma.$$

Similarly, the others are

$$a\cos\lambda'\cos\alpha+b\cos\mu'\cos\beta+c\cos\nu'\cos\gamma,$$
$$a\cos\lambda''\cos\alpha+b\cos\mu''\cos\beta+c\cos\nu''\cos\gamma;$$

and squaring and adding, we get the sum of the squares

$$a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma.$$

99. *The sum of the squares of the projections of three conjugate diameters on any fixed plane is constant.*

If d, d', d'' be the three diameters, θ, θ', θ'' the angles made by them with the perpendicular on the plane, the sum of the squares of the three projections is $d^2\sin^2\theta+d'^2\sin^2\theta'+d''^2\sin^2\theta''$, which is constant, since $d^2\cos^2\theta+d'^2\cos^2\theta'+d''^2\cos^2\theta''$ is constant by the last article, and $d^2+d'^2+d''^2$, by Art. 95.

100. *To find the locus of the intersection of three tangent planes at the extremities of three conjugate diameters.*

The equations of the three tangent planes are

$$\frac{x}{a}\cos\lambda+\frac{y}{b}\cos\mu+\frac{z}{c}\cos\nu=1,$$
$$\frac{x}{a}\cos\lambda'+\frac{y}{b}\cos\mu'+\frac{z}{c}\cos\nu'=1,$$
$$\frac{x}{a}\cos\lambda''+\frac{y}{b}\cos\mu''+\frac{z}{c}\cos\nu''=1.$$

Squaring and adding, we get for the equation of the locus

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3.$$

101. *To find the lengths of the axes of the section made by any plane passing through the centre.*

We can readily form the quadratic, whose roots are the reciprocals of the squares of the axes, since we are given the sum and the product of these quantities. Let α, β, γ be the

angles which a perpendicular to the given plane makes with the axes, R the intercept by the surface on this perpendicular; then we have (Art. 92)

$$\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{R^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

whence $\frac{1}{a'^2}+\frac{1}{b'^2}=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{\cos^2\alpha}{a^2}-\frac{\cos^2\beta}{b^2}-\frac{\cos^2\gamma}{c^2}\right),$

while (Art. 96) $\frac{1}{a'^2b'^2}=\frac{p^2}{a^2b^2c^2}=\frac{\cos^2\alpha}{b^2c^2}+\frac{\cos^2\beta}{c^2a^2}+\frac{\cos^2\gamma}{a^2b^2}.$

The quadratic required is therefore

$$\frac{1}{r^4}-\frac{1}{r^2}\left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right)+\frac{\cos^2\alpha}{b^2c^2}+\frac{\cos^2\beta}{c^2a^2}+\frac{\cos^2\gamma}{a^2b^2}=0.$$

This quadratic may also be written in the form

$$\frac{a^2\cos^2\alpha}{a^2-r^2}+\frac{b^2\cos^2\beta}{b^2-r^2}+\frac{c^2\cos^2\gamma}{c^2-r^2}=0.$$

This equation may be otherwise obtained from the principles explained in the next article.

102. *Through a given radius OR of a central quadric we can in general draw one section of which OR shall be an axis.*

Describe a sphere with OR as radius, and let a cone be drawn having the centre as vertex and passing through the intersection of the surface and the sphere, and let a tangent plane to the cone be drawn through the radius OR; then OR will be an axis of the section by that plane. For in it OR is equal to the next consecutive radius (both being radii of the same sphere) and is therefore a maximum or minimum; or, again, the tangent line at R to the section is perpendicular to OR, since it is also in the tangent plane to the sphere. OR is therefore an axis of the section.

The equation of the cone can at once be formed by subtracting one from the other, the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1,$$

when we get

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0.$$

If then any plane $x\cos\alpha+y\cos\beta+z\cos\gamma$ have an axis in length $=r$, it must touch this cone, and the condition that it should touch it, is (Art. 90)

$$\frac{a^2\cos^2\alpha}{a^2-r^2}+\frac{b^2\cos^2\beta}{b^2-r^2}+\frac{c^2\cos^2\gamma}{c^2-r^2}=0,$$

which is the equation found in the last article.

In like manner we can *find the axes of any central section of a quadric given by an equation of the form*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1.$$

The cone of intersection of this quadric with any sphere

$$\lambda(x^2+y^2+z^2)=1$$

is $(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy=0$, and we see, as before, that if λ be the reciprocal of the square of an axis of the section by the plane $x\cos\alpha+y\cos\beta+z\cos\gamma$, this plane must touch the cone whose equation has just been given. The condition that the plane should touch this cone (Art. 79) may be written

$$\begin{vmatrix} a-\lambda, & h, & g, & \cos\alpha \\ h, & b-\lambda, & f, & \cos\beta \\ g, & f, & c-\lambda, & \cos\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma, & \end{vmatrix}=0,$$

which expanded is

$$\lambda^2-\lambda\,\{(b+c)\cos^2\alpha+(c+a)\cos^2\beta+(a+b)\cos^2\gamma$$
$$-2f\cos\beta\cos\gamma-2g\cos\gamma\cos\alpha-2h\cos\alpha\cos\beta\}$$
$$+(bc-f^2)\cos^2\alpha+(ca-g^2)\cos^2\beta+(ab-h^2)\cos^2\gamma$$
$$+2(gh-af)\cos\beta\cos\gamma+2(hf-bg)\cos\gamma\cos\alpha$$
$$+2(fg-ch)\cos\alpha\cos\beta=0.$$

[Ex. 1. The principal axes of the section are the intersections of the plane with the cone

$$\begin{vmatrix} x, & y, & z \\ U_1, & U_2, & U_3 \\ \cos\alpha, & \cos\beta, & \cos\gamma \end{vmatrix}=0.$$

Ex. 2. If S be the area of a central section, S' the area of a parallel section, p' the central perpendicular on its plane and p the central perpendicular on the parallel tangent plane, prove

$$\frac{S'}{S}=1-\frac{p'^2}{p^2}.$$

Hence we can *find the axes and area of any section.*

Ex. 3. Show how to determine whether the section by any plane whose equation is given, is an ellipse, parabola or hyperbola.

Ex. 4. The condition that the quadratic for the axes may have equal roots may be written

$$\lambda a\sqrt{b^2 - c^2} \pm \mu b\sqrt{c^2 - a^2} \pm \nu c\sqrt{a^2 - b^2} = 0.$$

This condition is equivalent to the condition that the section may pass through *one* of the circular points at infinity in its plane. If it passes through *two* the section is a circle, the condition being λ or μ or $\nu = 0$. These may be proved by considering the equation determining the axes of a conic.]

Circular Sections.

103. We proceed to investigate whether it is possible to draw a plane which shall cut a given ellipsoid in a circle. As it has been already proved (Art. 73) that all parallel sections are similar curves, it is sufficient to consider sections made by planes through the centre. Imagine that any central section is a circle with radius r, and conceive a concentric sphere described with the same radius. Then we have just seen that

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0$$

represents a cone having the centre for its vertex and passing through the intersection of the quadric and the sphere. But if the surfaces have a plane section common, this equation must necessarily represent two planes, which cannot take place unless the coefficient of either x^2, y^2, or z^2 vanish. The plane section must therefore pass through one or other of the three axes. Suppose for example we take $r = b$, the coefficient of y vanishes, and there remains

$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = 0,$$

which represents two planes of circular section passing through the axis of y.

The two planes are easily constructed by drawing in the plane of xz a semi-diameter equal to b. Then the plane containing the axis of y, and either of the semi-diameters which can be so drawn, is a plane of circular section.

In like manner, two planes can be drawn through each of the other axes, but in the case of the ellipsoid these planes

will be imaginary; since we evidently cannot draw in the plane of xy a semi-diameter $= c$, the least semi-diameter in that section being $= b$; nor, again, in the plane of yz a semi-diameter $= a$, the greatest in that section being $= b$.

In the case of the hyperboloid of one sheet, c^2 is negative and the sections through a are those which are real. In the hyperboloid of two sheets, where both b^2 and c^2 are negative, if we take $r^2 = -c^2$ (b^2 being less than c^2), we get the two real sections,

$$x^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+y^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0.$$

These two real planes through the centre do not meet the surface, but parallel planes do meet it in circles. In all cases it will be observed that we have only two real central planes of circular section, the series of planes parallel to each of which afford two different systems of circular sections.

104. Any two surfaces whose coefficients of x^2, y^2, z^2 differ only by a constant, have the same planes of circular section. Thus

$Ax^2+By^2+Cz^2=1$, and $(A+H)x^2+(B+H)y^2+(C+H)z^2=1$ have the same planes of circular section, as easily appears from the formula in the last article.

The same thing appears by throwing the two equations into the form

$$\frac{1}{\rho^2}=A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma,$$

$$\frac{1}{\rho^2}=A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma+H,$$

from which it appears that the difference of the squares of the reciprocals of the corresponding radii vectores of the two surfaces is constant. If then in any section the radius vector of the one surface be constant, so must also the radius vector of the other. The same consideration shows that any plane cuts both in sections having the same axes, since the maximum or minimum value of the radius vector will in each correspond to the same values of α, β, γ.

Circular sections of a cone are the same as those of a hyperboloid to which it is asymptotic.

105. *Any two circular sections of opposite systems lie on the same sphere.*

The two planes of section are parallel each to one of the planes represented by

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0,$$

where $r=a$, b, or c.

Now since the equation of two planes agrees with the equation of two parallel planes as far as terms of the second degree are concerned, the equation of the two planes must be of the form

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)+u_1=0,$$

where u_1 represents some plane. If then we subtract this from the equation of the surface, which every point on the section must also satisfy, we get

$$\frac{1}{r^2}(x^2+y^2+z^2)-u_1=1,$$

which represents a sphere.

106. All parallel sections are, as we have seen, similar. If now we draw a series of planes parallel to circular sections, the extreme one will be the parallel tangent plane which must meet the surface in an infinitely small circle. Its point of contact is called an *umbilic*. Some properties of these points will be mentioned afterwards. The coordinates of the real umbilics are easily found. If $\lambda x+\nu z=0$ is a plane through the centre, any parallel plane is $\lambda x+\nu z+k=0$; if this touches the quadric, $a^2\lambda^2+c^2\nu^2=k^2$ and the points of contact are given by $x=\frac{\lambda a^2}{k}$, $z=\frac{\nu c^2}{k}$, whence, by Art. 103,

$$\frac{x^2}{a^2}=\frac{a^2-b^2}{a^2-c^2};\ \text{similarly}\ \frac{z^2}{c^2}=\frac{b^2-c^2}{a^2-c^2}.$$

There are accordingly in the case of the ellipsoid four real

umbilics in the plane of xz, and four imaginary in each of the other principal planes.

[The hyperboloid of one sheet has no real umbilics. The hyperboloid of two sheets has four real umbilics, two on each sheet.]

Rectilinear Generators.

107. We have seen that when the central section is an ellipse all parallel sections are similar ellipses, and the section by a tangent plane is an infinitely small similar ellipse. In like manner when the central section is a hyperbola, the section by any parallel plane is a similar hyperbola, and *that by the tangent plane reduces itself to a pair of right lines parallel to the asymptotes of the central hyperbola.** Thus if the equation referred to any conjugate diameters be

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}-\frac{z^2}{c'^2}=1,$$

and we consider the section made by any plane parallel to the plane of xz ($y=\beta$), its equation is

$$\frac{x^2}{a'^2}-\frac{z^2}{c'^2}=1-\frac{\beta^2}{b'^2}.$$

And it is evident that the value $\beta=b'$ reduces the section to a pair of right lines. The hyperboloid of one sheet is the only central surface on which such right lines exist,† since if we had the equation

$$\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1+\frac{z^2}{c'^2},$$

the right-hand side of the equation could not vanish for any real value of x. It is also geometrically evident that a right line cannot exist either on an ellipsoid, which is a closed surface, or on a hyperboloid of two sheets, no part of which, as we saw, lies in the space included between several systems of

* Cf. Art. 80*a*.

† It will be understood that the remarks in the text apply only to *real* right lines; *every* quadric surface has upon it an infinity of right lines, real or imaginary, and (not being a cone) it is a skew surface. See footnote, Art. 112.

two parallel planes, while any right line will of course in general intersect them all.

108. Throwing the equation of the hyperboloid of one sheet into the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2},$$

it is evident that the intersection of the two planes

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right),\ \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)$$

lies on the surface; and by giving different values to λ we get a system of right lines lying in the surface; while, again, we get another system by considering the intersection of the planes

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 + \frac{y}{b}\right),\ \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b}.$$

What has been just said may be stated more generally as follows: If α, β, γ, δ represent four planes, then the equation $\alpha\gamma = \beta\delta$ represents a hyperboloid of one sheet, which may be generated as the locus of the system of right lines $\alpha = \lambda\beta$, $\lambda\gamma = \delta$, or of the system $\alpha = \lambda\delta$, $\lambda\gamma = \beta$. [The right lines are therefore called *rectilinear generators*.]

Considering four lines in either system as $\alpha = \lambda\beta$, $\lambda\gamma = \delta$, λ having four different values we have two pencils of planes which we see by Art. 39 are equianharmonic; hence *the hyperboloid of one sheet may be regarded as the locus of lines of intersection of two homographic pencils of planes.*

In the case of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the lines may be also expressed by the equations

$$\frac{x}{a} = \frac{z}{c}\cos\theta \mp \sin\theta,\ \frac{y}{b} = \frac{z}{c}\sin\theta \pm \cos\theta.$$

[The condition that the right line

$$\frac{x - x'}{\lambda} = \frac{y - y'}{\mu} = \frac{z - z'}{\nu}$$

may lie altogether on a surface $U = 0$ found (cf. Art. 80*a*) by substituting

$x' + \rho\lambda$, $y' + \rho\mu$, $z' + \rho\nu$ in U and equating to zero the coefficients of powers of ρ and the absolute term. For the general quadric this gives

$$U' = 0, \ \lambda\frac{dU'}{dx'} + \mu\frac{dU'}{dy'} + \nu\frac{dU'}{dz'} = 0, \text{ and } (a, b, c, f, g, h)\ (\lambda, \mu, \nu)^2 = 0.$$

The second equation expresses that *the line lies in the tangent plane at* x', y', z', the third that it lies in a cone with its vertex at x', y', z', and parallel to the asymptotic cone. *Thus all the generators of a central quadric are parallel to the generators of the asymptotic cone.*

If we consider each of the typical quadrics (Chap. V.) we can easily prove analytically that the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid have no real generators, and that the other quadrics have two real generators at every point; for if x', y', z' are real λ, μ, ν are imaginary in one case and real in the other.]

109. *Any two lines belonging to opposite systems lie in the same plane.*

Consider the two lines

$$\alpha - \lambda\beta, \ \lambda\gamma - \delta,$$
$$\alpha - \lambda'\delta, \ \lambda'\gamma - \beta.$$

Then it is evident that the plane $\alpha - \lambda\beta + \lambda\lambda'\gamma - \lambda'\delta$ contains both, since it can be written in either of the forms

$$\alpha - \lambda\beta + \lambda'(\lambda\gamma - \delta), \ \alpha - \lambda'\delta + \lambda\ (\lambda'\gamma - \beta).$$

It is evident in like manner that *no two lines belonging to the same system lie in the same plane.* In fact, no plane of the form $(\alpha - \lambda\beta) + k\ (\lambda\gamma - \delta)$ can ever be identical with $(\alpha - \lambda'\beta) + k'\ (\lambda'\gamma - \delta)$ if λ and λ' are different. In the same way we see that both the lines

$$\frac{x}{a} = \frac{z}{c}\cos\theta - \sin\theta, \ \frac{y}{b} = \frac{z}{c}\sin\theta + \cos\theta,$$

$$\frac{x}{a} = \frac{z}{c}\cos\phi + \sin\phi, \ \frac{y}{b} = \frac{z}{c}\sin\phi - \cos\phi,$$

which belong to different systems, lie in the plane

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta + \phi) + \frac{y}{b}\sin\tfrac{1}{2}(\theta + \phi) = \frac{z}{c}\cos\tfrac{1}{2}(\theta - \phi) - \sin\tfrac{1}{2}(\theta - \phi).$$

Now this plane is parallel to the second line of the first system

$$\frac{x}{a} = \frac{z}{c}\cos\phi - \sin\phi, \ \frac{y}{b} = \frac{z}{c}\sin\phi + \cos\phi,$$

but it does not pass through it, for the equation of a parallel plane through this line will be found to be

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\frac{z}{c}\cos\tfrac{1}{2}(\theta-\phi)+\sin\tfrac{1}{2}(\theta-\phi),$$

which differs in the absolute term from the equation of the plane through the first line.

[That the generators fall into two systems, each generator of one system meeting no generator of the same system and meeting every generator of the opposite system, may also be seen as follows. The tangent plane at A cuts the quadric in two generators meeting the tangent plane at B in C and D. Then BC and BD are the generators at B. Thus the generator AC meets the generator BC and the generator AD meets the generator BD. AC cannot meet BD, and AD cannot meet BC, for if either of these happened the four generators would lie in a plane, which would therefore meet the quadric in a curve of the fourth order. We have thus a twisted quadrilateral $ACBD$ composed of generators, and the plane joining any three of its vertices is the tangent plane at one of them. The diagonals AB and CD are polar lines (Art. 65).

Ex. 1. Four fixed generators of one system are joined to a variable generator of the opposite system; prove that the four planes have a constant anharmonic ratio (cf. Art. 108). This may be proved from the anharmonic property of conics by considering the section by any plane.

Ex. 2. If a quadric passes through a fixed quadrilateral in space the locus of its centre is the line joining the middle points of the diagonals.]

110. We have seen that any tangent plane to the hyperboloid meets the surface in two right lines intersecting in the point of contact, and of course touches the surface in no other point. If through one of these right lines we draw any *other* plane, we have just seen that it will meet the surface in a new right line, and this new plane will touch the surface in the point where these two lines intersect. Conversely, the tangent plane to the surface at any point on a given right line in the surface will contain the right line, but the tangent plane will in general be different for every point of the right line. Thus, take the surface $x\phi=y\psi$, where the line xy lies on the surface, and ϕ and ψ represent planes (though the demonstration would equally hold if they were functions of any higher degree). Then using the equation of the tangent plane

$$(x-x')\,U_1'+(y-y')\,U_2'+(z-z')\,U_3'=0,$$

and seeking the tangent at the point $x=0$, $y=0$, $z=z'$, we find $x\phi'=y\psi'$, where ϕ' and ψ' are what ϕ and ψ become on

substituting these coordinates. And this plane will vary as z' varies.

It is easy also to deduce from this that *the anharmonic ratio of four tangent planes passing through a right line in the surface is equal to that of their four points of contact along the line.* [This is a special case of the theorem that the anharmonic ratio of four planes meeting in a line is equal to the anharmonic ratio of their four (collinear) poles.]

All this is different in the case of the cone. Here every tangent plane meets the surface in two coincident right lines. The tangent plane then at every point of this right line is the same, and the plane touches the surface along the whole length of the line.

And generally, if the equation of a surface be of the form

$$x\phi + y^2\psi = 0,$$

it is seen precisely, as above, that the tangent plane at every point of the line xy is $x = 0$.

[Ex. If two quadrics have a common generator, the tangent planes at two points on the generator coincide.]

111. It was proved (Art. 107) that the two lines in which the tangent plane cuts a hyperboloid are parallel to the asymptotes of the parallel central section; but these asymptotes are evidently edges of the asymptotic cone to the surface. Hence every right line which can lie on a hyperboloid is parallel to some one of the edges of the asymptotic cone (cf. Art. 108). It follows also that *three of these lines (unless two of them are parallel) cannot all be parallel to the same plane;* since, if they were, a parallel plane would cut the asymptotic cone in three edges, which is impossible, the cone being only of the second degree.

112. We have seen that any line of the first system meets all the lines of the second system. Conversely, the surface may be conceived as generated by the motion of a right line which always meets a certain number of fixed right lines.*

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* A surface generated by the motion of a right line is called a *ruled* surface. If every generating line is intersected by the next consecutive one, the

Let us remark, in the first place, that when we are seeking the surface generated by the motion of a right line, it is necessary that the motion of the right line should be regulated by *three* conditions. In fact, since the equations of a right line include four constants, four conditions would absolutely determine the position of a right line. When we are given one condition less, the position of the line is not determined, but it is so far limited that the line will always lie on a certain surface-locus, whose equation can be found as follows: Write down the general equations of a right line $x=mz+p$, $y=nz+q$; then the conditions of the problem establish three relations between the constants m, n, p, q. And combining these three relations with the two equations of the right line, we have five equations from which we can eliminate the four quantities m, n, p, q; and the resulting equation in x, y, z will be the equation of the locus required. Or, again, we may write the equations of the line in the form

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma},$$

then the three conditions give three relations between the constants $x', y', z', \alpha, \beta, \gamma$, and if between these we eliminate α, β, γ, the resulting equation in x', y', z' is the equation of the required locus, since $x'y'z'$ may be any point on the line.

We see then, that it is a determinate problem to find the surface generated by a right line which moves so as always to meet *three* fixed right lines.* For, expressing, by Art. 43, the condition that the movable right line shall meet each of the fixed lines, we obtain the three necessary relations between m, n, p, q. Geometrically also we can see that the motion of the line is completely regulated by the given conditions. For a line would be completely determined if it were constrained to pass through a given point and to meet two fixed lines,

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surface is called a *developable* or *torse*. If not, it is called a *skew* surface or *scroll*. The hyperboloid of one sheet, and indeed every quadric surface (not being a cone or cylinder) belongs to the latter class; the cone and cylinder to the former.

* Or three fixed curves of any kind

since we need only draw planes through the given point and each of the fixed lines, when the intersection of these planes would determine the line required. If, then, a point through which the line is to pass, itself moves along a third fixed line, we have a determinate series of right lines, the assemblage of which forms a surface-locus.

113. Let us then solve the problem suggested by the last article, viz. *to find the surface generated by a right line which always meets three fixed right lines, no two of which are in the same plane.* In order that the work may be shortened as much as possible, let us first examine what choice of axes we must make in order to give the equations of the fixed right lines the simplest form.

And it occurs at once that we ought to take the axes, one parallel to each of the three given right lines.* The only question then is, where the origin can most symmetrically be placed. Suppose now, that through each of the three right lines we draw planes parallel to the other two, we get thus three pairs of parallel planes forming a parallelepiped, of which the given lines will be edges. And if through the centre of this parallelepiped we draw lines parallel to these edges, we shall have the most symmetrical axes. Let then the equations of the three pairs of planes be

$$x = \pm a,\ y = \pm b,\ z = \pm c,$$

then the equations of the three fixed right lines will be

$$y = b,\ z = -c\ ;\ z = c,\ x = -a\ ;\ x = a,\ y = -b.$$

The equations of any line meeting the first two fixed lines are

$$z + c = \lambda\ (y - b),\quad z - c = \mu\ (x + a),$$

which will intersect the third if $c + \mu a + \lambda b = 0$; or replacing for λ and μ their values,

$$c\ (x + a)\ (y - b) + a\ (z - c)\ (y - b) + b\ (z + c)\ (x + a),$$

which reduced is

$$ayz + bzx + cxy + abc = 0.$$

* We could not do this indeed if the three given right lines happened to be all parallel to the same plane. This case will be considered in the next section. It will not occur when the locus is a hyperboloid of one sheet, see Art. 111.

This represents a hyperboloid of one sheet, since it represents a central quadric, and is known to be a ruled surface. The problem might otherwise be solved thus:

Assuming for the equations of the movable line

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma},$$

the following three conditions are obtained by expressing that this intersects each of the fixed lines,

$$\frac{y'-b}{\cos\beta}=\frac{z'+c}{\cos\gamma},\ \frac{z'-c}{\cos\gamma}=\frac{x'+a}{\cos\alpha},\ \frac{x'-a}{\cos\alpha}=\frac{y'+b}{\cos\beta}.$$

We can eliminate α, β, γ by multiplying the equations together, and get for the equation of the locus,

$$(x-a)\ (y-b)\ (z-c)=(x+a)\ (y+b)\ (z+c),$$

which reduces to $ayz+bzx+cxy+abc=0$ the same equation as before.

The last written form of the equation expresses that this hyperboloid is the locus of a point, the product of whose distances from three concurrent faces of a parallelepiped is equal to the product of its distances from the three opposite faces.

The following is another general solution of the same problem: Let the first two lines be the intersections of the planes α, β; γ, δ; then the equations of the third can be expressed in the form $\alpha=A\gamma+B\delta$, $\beta=C\gamma+D\delta$. The movable line, since it meets the first two lines, can be expressed by two equations of the form $\alpha=\lambda\beta$, $\gamma=\mu\delta$. Substituting these values in the equations of the third line, we find the condition that it and the movable line should intersect, viz.

$$A\mu+B=\lambda\ (C\mu+D).$$

And eliminating λ and μ between this and the equations of the movable line, we get for the equation of the locus,

$$\beta\ (A\gamma+B\delta)=\alpha\ (C\gamma+D\delta).$$

A third general solution is as follows: taking $(p_1, q_1, r_1, s_1, t_1, u_1)$, $(p_2, \ldots)$, $(p_3, \ldots)$ as the six coordinates of the given lines respectively, and writing for shortness (pqr) to denote the determinant $p_1(q_2r_3-q_3r_2)+$ &c., and so in other cases, then it can be shown (Art. 53c) that the equation of the hyperboloid passing through the three given lines is

$$\begin{aligned}&(ptu)x^2+(qus)y^2+(rst)z^2+(pqr)w^2\\&+[(pqt)-(rpu)]xw+[(qst)+(rus)]yz\\&+[(qru)-(pqs)]yw+[(rtu)+(pst)]zx\\&+[(rps)-(qrt)]zw+[(pus)+(qtu)]xy=0.\end{aligned}$$

114. *Four right lines belonging to one system cut all lines belonging to the other system in a constant anharmonic ratio.*

For through the four lines and through any line which meets them all we can draw four planes; and therefore any other line which meets the four lines will be divided in a constant anharmonic ratio (Art. 39).

Conversely, if two non-intersecting lines are divided *homographically* in a series of points, that is to say, so that the anharmonic ratio of any four points on one line is equal to that of the corresponding points on the other, then the lines joining corresponding points will be generators of a hyperboloid of one sheet.

If the equations of the two given lines are $\alpha=\beta=0$, $\gamma=\delta=0$, the equations of any line meeting them both may be written $\alpha=\lambda\beta$, $\gamma=\mu\delta$, and if this line joins corresponding points, λ and μ are connected by an equation of the form $a\lambda\mu+b\lambda+c\mu+d=0$ (*Conics*, Art. 331). If we eliminate λ and μ, we obtain the equation of the locus in the form

$$a\alpha\gamma+b\alpha\delta+c\beta\gamma+d\beta\delta=0.$$

[The tangent cone from a point to a quadric is the envelope of the planes joining P to the generators of the quadric. Hence.

Ex. 1. The conical projections of the generators from any point P on a plane are tangents to the conic in which the plane meets the tangent cone from P.

Ex. 2. Similar theorems hold for the conical projections on any surface; for example the conical projections of the generators on to another quadric are a system of conics touching a twisted quartic (the intersection of the cone and the second quadric).

Ex. 3. If the coordinates be transformed by any linear substitution a quadric and its generators are transformed into a quadric and its generators.

Ex. 4. Given three generators of the same system of a hyperboloid, the hyperboloid we have seen, is determined. Show how to construct geometrically the centre, the polar plane of a given point, and the pole of a given plane.

Ex. 5. If two systems of four non-intersecting right lines have 15 points in common, they have another point in common.]

Non-central Surfaces.

115. The reader is recommended to work out for himself the properties of paraboloids which are analogous to the results of the preceding articles of this chapter. In particular he may prove * the following :—

[We define a diameter as any line drawn parallel to the axis of the paraboloid. Let S be the conic (ellipse or hyperbola) in which a fixed plane conjugate to the diameter through a point P on the paraboloid (i.e. parallel to the tangent plane at P) cuts the paraboloid. Then

(1) The parameter at P of any diametral section through P is proportional to the square of that radius of S in which the section cuts the plane of S. †

Again, two diametral planes are conjugate if the two radii of S in which they meet the plane of S are conjugate radii. Thus homogeneous relations between conjugate radii of a conic lead to the same relations between the parameters of diametral sections of a paraboloid at a fixed point.]

(2) The sum or difference of the principal parameters of any two conjugate diametral sections of a paraboloid is constant according as it is elliptic or hyperbolic.

(3) The sum or difference of the parameters of any two conjugate diametral sections at a given point of a paraboloid is constant, according as it is elliptic or hyperbolic.

(4) If from the extremity of any diameter of a paraboloid a line of constant length be measured along the diameter, and a fixed conjugate plane drawn cutting the paraboloid, the volume of the parallelepiped formed by any two conjugate diameters of the section and this line is constant.

We proceed to determine *the circular section of the paraboloid* given by the equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \frac{2z}{c}.$$

Consider a circular section through the origin, and describe a sphere through it having, at the origin, the same tangent plane (z) as the paraboloid; then (Art. 61) the equation of the sphere must be of the form

$$x^2 + y^2 + z^2 = 2nz.$$

* See Allman, On some Properties of the Paraboloids, *Quarterly Journal of Pure and Applied Mathematics*, 1874.

† [When the equation of a parabola is expressed in the form $y^2 = px$, p is called the parameter at the origin; if the axes are rectangular, p is the *principal* parameter (Conics, Art. 205). Parallel sections have the same parameter.]

And the cone of intersection of this sphere with the paraboloid is

$$x^2\left(1-\frac{cn}{a^2}\right)+y^2\left(1\mp\frac{cn}{b^2}\right)+z^2=0.$$

This will represent two planes if one of the terms vanishes. It will represent *two real planes in the case of the elliptic paraboloid*, if we take $\frac{cn}{a^2}=1$, for the equation then becomes $b^2z^2=(a^2-b^2)y^2$. But *in the case of the hyperbolic paraboloid there is no real circular section*, since the same substitution would make the equation of the two planes take the imaginary form $b^2z^2+(a^2+b^2)\,y^2=0$.

Indeed, it can be proved that *no section of the hyperbolic paraboloid can be a closed curve*, for if we take its intersection with any plane $z=\alpha x+\beta y+\gamma$, the projection on the plane of xy is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2(\alpha x+\beta y+\gamma)}{c}$ which is necessarily a hyperbola. [Cf. also Art. 87, V.]

116. From the general theory explained in Art. 108, it is plain that the hyperbolic paraboloid may also have right lines lying altogether in the surface. For the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ (Art. 87) is included in the general form $\alpha\gamma=\beta\delta$, and the surface contains the two systems of right lines

$$\frac{x}{a}\pm\frac{y}{b}=\lambda,\ \lambda\left(\frac{x}{a}\mp\frac{y}{b}\right)=\frac{z}{c}.$$

The first equation shows that *every right line on the surface must be parallel to one or other of the two fixed planes* $\frac{x}{a}\pm\frac{y}{b}=0$; this is the fundamental difference between right lines on the paraboloid and on the hyperboloid (see Art. 111). [If the general equation be used, the terms of the highest degree break into two linear factors LM, and all generators are parallel to one of the planes $L=0$, $M=0$.]

It is proved, as in Art. 109, that any line of one system meets every line of the other system, while no two lines of the same system can intersect.

We give now the investigation of the converse problem, viz. *to find the surface generated by a right line which always meets three fixed lines which are all parallel to the same plane.* Let the plane to which all are parallel be taken for the plane of xy, any line which meets all three for the axis of z, and let the axis of x and y be taken parallel to two of the fixed lines. Then their equations are

$$x=0,\ z=a\ ;\ y=0,\ z=b\ ;\ x=my,\ z=c.$$

The equations of any line meeting the first two fixed lines are

$$x=\lambda\ (z-a),\ y=\mu\ (z-b),$$

which will intersect the third if

$$\lambda\ (c-a)=m\mu\ (c-b),$$

and the equation of the locus is

$$(a-c)\ x\ (z-b)=m(b-c)\ y\ (z-a),$$

which represents a hyperbolic paraboloid, since the terms of highest degree break up into two real factors.

In like manner we might investigate the surface generated by a right line which meets *two* fixed lines and is always parallel to a fixed plane. Let it meet the lines

$$x=0,\ z=a\ ;\ y=0,\ z=\ -a,$$

and be parallel to the plane

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p.$$

Then the equations of the line are

$$x=\lambda\ (z-a),\ y=\mu\ (z+a),$$

which will be parallel to the given plane if

$$\cos\gamma+\lambda\cos\alpha+\mu\cos\beta=0.$$

The equation of the required locus is therefore

$$(z^2-a^2)\cos\gamma+x\ (z+a)\cos\alpha+y\ (z-a)\cos\beta=0,$$

which is a hyperbolic paraboloid, since the terms of the second degree break up into two real factors.

A hyperbolic paraboloid is the limit of the hyperboloid of one sheet, when the generator in one of its positions may lie altogether at infinity.

We have seen (Art. 107) that a plane is a tangent to a surface of the second degree when it meets it in two real or imaginary lines; and since a paraboloid may be written $xy=kz$, it is met by the plane at infinity in two real or

imaginary lines. Hence *a paraboloid is always touched by the plane at infinity.*

[Ex. 1. If a hyperbolic paraboloid passes through the sides of a quadrilateral $ABCD$ in space, it bisects the volume of the tetrahedron where vertices are A, B, C, and D. For AB, CD are parallel to a fixed plane and AD, BC are parallel to a second fixed plane. Let L be any plane parallel to the former plane, and let it meet AC in P, AD in Q, BD in R, and BC in S. The line QS generates the surface as L varies; also $PQRS$ is a parallelogram and therefore the triangle $QRS = QPS$; whence the theorem follows.

Ex. 2. Prove that normals to any quadric along a generator generate a hyperbolic paraboloid.

Ex. 3. Show that the generators of hyperbolic paraboloid may be orthogonally projected into a system of parallelograms on a plane. Prove that the orthogonal projections on the planes of x or y envelop parabolas.]

117. In the case of the hyperbolic paraboloid *any three right lines of one system cut all the right lines of the other in a constant ratio.* For since the generators are all parallel to the same plane, we can draw, through any three generators, parallels to that plane, and all right lines which meet three parallel planes are cut by them in a constant ratio.

Conversely, if two finite non-intersecting lines be divided, each into the same number of equal parts, the lines joining corresponding points will be generators of a hyperbolic paraboloid. By doing this with threads, the form of this surface can be readily exhibited to the eye.

To prove this directly, let the line which joins two corresponding extremities of the given lines be the axis of z; let the axes of x and y be taken parallel to the given lines, and let the plane of xy be half-way between them. Let the lengths of the given lines be a and b, and $2c$ the distance between them, then the coordinates of two corresponding points are

$$z = c, \quad x = \mu a, \; y = 0,$$
$$z = -c, \; x = 0, \quad y = \mu b,$$

and the equations of the line joining these points are

$$\frac{x}{a} + \frac{y}{b} = \mu, \; 2cx - \mu az = \mu ac,$$

whence, eliminating μ, the equation of the locus is

$$2cx = a(z+c)\left(\frac{x}{a} + \frac{y}{b}\right),$$

which represents a hyperbolic paraboloid.

Surfaces of Revolution.

118. Let it be required to find the conditions that the general equation should represent a surface of revolution. In this case the equation can be reduced (see Art. 84), if the surface be central to the form $\frac{x^2}{a^2}+\frac{y^2}{a^2}\pm\frac{z^2}{c^2}=\pm 1$, and if the surface be non-central to the form $\frac{x^2}{a^2}+\frac{y^2}{a^2}=\frac{2z}{c}$. In either case then, when the highest terms are transformed so as to become the sum of squares of three rectangular coordinates, the coefficients of two of those squares are equal. It would appear then that the required condition could be at once obtained by forming the condition that the discriminating cubic should have equal roots. Since, however, the roots of the discriminating cubic are always real, its discriminant can be expressed as the sum of squares (see *Higher Algebra*, Art. 53), and will not vanish (the coefficients of the given equation being supposed to be real) unless *two* conditions are fulfilled, which can be obtained more easily by the following process. We want to find whether it is possible so to transform the equation as to have

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=A\ (X^2+Y^2)+CZ^2,$$

but we have (Art. 19)

$$x^2+y^2+z^2=X^2+Y^2+Z^2.$$

It is manifest then that by taking $\lambda=A$, we should have the following quantity a perfect square:

$$(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)-\lambda\ (x^2+y^2+z^2),$$

and it is required to find the conditions that this should be possible.

Now it is easy to see that when

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy$$

is a perfect square, the six following conditions are fulfilled:

$$BC=F^2,\quad CA=G^2,\quad AB=H^2,$$
$$AF=GH,\ BG=HF,\ CH=FG;$$

the three former of which are included in the three latter. In the present case then these latter three equations are

$$(a-\lambda)\ f=gh,\ (b-\lambda)\ g=hf,\ (c-\lambda)\ h=fg.$$

Solving for λ from each of these equations we see that the reduction is impossible unless the coefficients of the given equation be connected by the two relations

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}.$$

If these relations be fulfilled, and if we substitute any of these common values for λ in the function

$$(a-\lambda)x^2 + (b-\lambda)y^2 + (c-\lambda)z^2 + 2fyz + 2gzx + 2hxy,$$

it becomes, as it ought, a perfect square, viz.

$$fgh\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right)^2 = (C - A)Z^2,$$

and since the plane $Z = 0$ represents a plane perpendicular to the axis of revolution of the surface, it follows that

$$\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$$

represents a plane perpendicular to that axis.

In the special case where the common values vanish which have been just found for λ, the highest terms in the given equation form a perfect square, and the equation represents either a parabolic cylinder or two parallel planes (see IV. and V., Art. 87). These are limiting cases of surfaces of revolution, the axis of revolution in the latter case being any line perpendicular to both planes. The parabolic cylinder is the limit of the surface generated by the revolution of an ellipse round its minor axis, when that axis passes to infinity.

[Ex. If the general equation represents a surface of revolution, prove that the equations of the axis of revolution are

$$\frac{U_1}{\frac{a}{f} + \frac{h}{g} + \frac{g}{h}} = \frac{U_2}{\frac{h}{f} + \frac{b}{g} + \frac{f}{h}} = \frac{U_3}{\frac{g}{f} + \frac{f}{g} + \frac{c}{h}}.]$$

119. If one of the quantities f, g, h vanish, the surface cannot be of revolution unless a second also vanish. Suppose that we have f and g both $= 0$, the preceding conditions become

$$a - h\frac{g}{f} = b - h\frac{f}{g} = c,$$

from which, eliminating the indeterminate $\frac{f}{g}$, we get

$$(a-c)\ (b-c) = h^2.$$

This condition might also have been obtained at once by expressing that

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2hxy$$

should be a perfect square, and it is plain that we must have

$$\lambda=c\,;\ \ (a-c)\ (b-c)=h^2.$$

120. The preceding theory might also be obtained from the consideration that in a surface of revolution the problem of finding the principal planes becomes indeterminate. For since every section perpendicular to the axis of revolution is a circle, any system of parallel chords of one of these circles is bisected by the plane passing through the axis of revolution and through the diameter of the circle perpendicular to the chords, a plane which is perpendicular to the chords. It follows that *every* plane through the axis of revolution is a principal plane. Now the chords which are perpendicular to these diametral planes are given (Art. 72) by the equations

$$(a-\lambda)x+hy+gz=0,\ hx+(b-\lambda)y+fz=0,\ gx+fy+(c-\lambda)z=0,$$

which, when λ is one of the roots of the discriminating cubic, represent three planes meeting in one of the right lines required. The problem then will not become indeterminate unless these equations all represent the same plane, for which we have the conditions

$$\frac{a-\lambda}{h}=\frac{h}{b-\lambda}=\frac{g}{f}\,;\ \ \frac{a-\lambda}{g}=\frac{h}{f}=\frac{g}{c-\lambda},$$

which, expanded, are the same as the conditions found already.

Loci.

121. We shall conclude this chapter by a few examples of the application of Algebraic Geometry to the *investigation of Loci.*

Ex. 1. To find the locus of a point whose shortest distances from two given non-intersecting right lines are equal.

If the equations of the lines are written in their general form, the solution of this is obtained immediately by the formula of Art. 14. We may get the result in a simple form by taking for the axis of z the shortest distance between the two lines, and choosing for the other axes the lines bisecting the

angle between parallels to the given lines through the point of bisection of this shortest distance, then their equations are of the form

$$z = c,\ y = mx;\quad z = -c,\ y = -mx,$$

and the conditions of the problem give

$$(z-c)^2 + \frac{(y-mx)^2}{1+m^2} = (z+c)^2 + \frac{(y+mx)^2}{1+m^2},$$

or
$$cz(1+m^2) + mxy = 0.$$

The locus is therefore a hyperbolic paraboloid.

If the shortest distances had been to each other in a given ratio, the locus would have been

$$\{(1+\lambda)z + (1-\lambda)c\}\{(1-\lambda)z + (1+\lambda)c\}$$
$$+ \frac{1}{1+m^2}\{(1+\lambda)y + (1-\lambda)mx\}\{(1-\lambda)y + (1+\lambda)mx\} = 0,$$

which represents a hyperboloid of one sheet.

Ex. 2. To find the locus of the middle points of all lines parallel to a fixed plane and terminated by two non-intersecting lines.

Take the plane $x = 0$ parallel to the fixed plane, and the plane $z = 0$ as in the last example parallel to the two lines and equidistant from them; then the equations of the lines are

$$z = c,\ y = mx + n;\quad z = -c,\ y = m'x + n'.$$

The locus is then evidently the right line which is the intersection of the planes

$$z = 0,\quad 2y = (m + m')x + (n + n').$$

Ex. 3. To find the surface of revolution generated by a right line turning round a fixed axis which it does not intersect.

Let the fixed line be the axis of z, and let any position of the other be $x = mz + n,\ y = m'z + n'$. Then since any point of the revolving line describes a circle in a plane parallel to that of xy, it follows that the value of $x^2 + y^2$ is the same for every point in such a plane section, and it is plain that the constant value expressed in terms of z is $(mz+n)^2 + (m'z+n')^2$. Hence the equation of the required surface is

$$x^2 + y^2 = (mz+n)^2 + (m'z+n')^2,$$

which represents a hyperboloid of revolution of one sheet.

Ex. 4. Two lines passing through the origin move each in a fixed plane remaining perpendicular to each other; to find the surface (necessarily a cone) generated by a right line also passing through the origin perpendicular to the other two.

Let the direction angles of the perpendiculars to the fixed planes be a, b, c; a', b', c'; and let those of the variable line be α, β, γ; then the direction cosines of the intersections with the fixed planes of a plane perpendicular to the variable line will (Art. 15) be proportional to

$$\cos\beta\cos c - \cos\gamma\cos b,\ \cos\gamma\cos a - \cos\alpha\cos c,\ \cos\alpha\cos b - \cos\beta\cos a,$$
$$\cos\beta\cos c' - \cos\gamma\cos b',\ \cos\gamma\cos a' - \cos\alpha\cos c',\ \cos\alpha\cos b' - \cos\beta\cos a',$$

and the condition that these should be perpendicular to each other is

$$\begin{vmatrix} x, & y, & z \\ \cos a, & \cos b, & \cos c \end{vmatrix}\begin{vmatrix} x, & y, & z \\ \cos a', & \cos b', & \cos c' \end{vmatrix} = 0, \text{ where } x = r\cos\alpha \ \&c.$$

which represents a cone of the second degree. [This relation expresses

$r^2 \cos A = pp'$, where r is distance from origin, A is the angle between the planes and p, p' are the perpendiculars on the two planes. This can be proved also by spherical geometry.]

Ex. 5. Two planes mutually perpendicular pass each through a fixed line; to find the surface generated by their line of intersection.

Take the axes as in Ex. 1. Then the equations of the planes are

$$\lambda (z - c) + y - mx = 0; \; \lambda' (z + c) + y + mx = 0,$$

which will be at right angles if $\lambda\lambda' + 1 - m^2 = 0$; and putting in for λ, λ' their values from the pair of equations we get

$$y^2 - m^2x^2 + (1 - m^2)(z^2 - c^2) = 0,$$

which represents a hyperboloid of one sheet. In either case, if the lines intersect, making $c = 0$, the locus reduces to a cone.

Ex. 5*a*. Both the hyperboloid of Ex. 5 and of Ex. 1 are such that two pairs of generators are perpendicular to the planes of circular sections. Such hyperboloids of one sheet have been called *orthogonal hyperboloids* (Schröter, *Crelle's Jour.*, Vol. 85).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ is orthogonal if } \frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2} = 0.$$

Ex 6. To find the locus of a point, whence three tangent lines, mutually at right angles, can be drawn to the quadric $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

If the equation were transformed so that these lines should become the axes of coordinates, the equation of the tangent cone would take the form $Ayz + Bzx + Cxy = 0$, since these three lines are edges of the cone. But the untransformed equation of the tangent cone is, see Art. 78,

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} - 1\right)^2.$$

And we have seen (Art. 82) that if this equation be transformed to any rectangular system of axes, the sum of the coefficients of x^2, y^2, and z^2 will be constant. We have only then to express the condition that this sum should vanish, when we obtain as equation of the required locus,

$$\frac{x^2}{a^2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{y^2}{b^2}\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + \frac{z^2}{c^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Ex. 7. The plane through the extremities of conjugate diameters of an ellipsoid envelopes the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = \frac{1}{3}$ and touches it in the centre of the section.

Ex. 8. The condition that the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ may admit of three generators mutually at right angles is found to be $\dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{c^2} = 0$.

Such hyperboloids have been called *equilateral hyperboloids* (Schröter, *Oberflachen zweiter Ordnung*, p. 197, 1880).

Ex. 9. To find the equation of the cone whose vertex is $x'y'z'$ and which stands on the conic in the plane of xy, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The equations of the line joining any point $\alpha\beta$ of the base to the vertex are

$$\alpha(z' - z) = z'x - x'z,\ \beta(z' - z) = z'y - y'z.$$

Substituting those values in the equation of base, we get for the required cone

$$\frac{(z'x - x'z)^2}{a^2} + \frac{(z'y - y'z)^2}{b^2} = (z' - z)^2.$$

The following method may be used in general to find the equation of the cone whose vertex is $x'y'z'w'$, and base the intersection of any two surfaces U, V. Substitute in each equation for x, $x + \lambda x'$, for y, $y + \lambda y'$, &c., and let the results be

$$U + \lambda\delta U + \frac{\lambda^2}{1.2}\delta^2 U + \&c. = 0,\ V + \lambda\delta V + \frac{\lambda^2}{1.2}\delta^2 V + \&c. = 0,$$

then the result of eliminating λ between these equations will be the equation of the required cone. For the points where the line joining $x'y'z'w'$ to $xyzw$ meets the surface U are got from the first of these two equations; those where the same line meets the surface V are got from the second; and when the eliminant of the two equations vanishes they have a common root, or the point $xyzw$ lies on a line passing through $x'y'z'w'$ and meeting the intersection of the surfaces.

Ex. 10. To find the equation of the cone whose vertex is the centre of an ellipsoid and base the section made by the polar of any point $x'y'z'$.

$$Ans.\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right)^2.$$

Ex. 11. To find the locus of points on the quadric $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the normals at which intersect the normal at the point $x'y'z'$.

Ans. The locus required is the intersection of the surface with the cone,

$$a^2(y'z - z'y)(x - x') + b^2(z'x - x'z)(y - y') + c^2(x'y - y'x)(z - z') = 0.$$

Ex. 12. To find the locus of the poles of the tangent planes of one quadric with respect to another.

We have only to express the condition that the polar of $x'y'z'w'$, with regard to the second quadric, should touch the first, and have therefore only to substitute U_1, U_2, U_3, U_4 for α, β, γ, δ in the condition given Art. 79. The locus is therefore a quadric.

Ex. 13. To find the cone generated by perpendiculars erected at the vertex of a given cone to its several tangent planes.

Let the cone be $Lx^2 + My^2 + Nz^2 = 0$, then any tangent plane is $Lx'x + My'y + Nz'z = 0$ the perpendicular to which through the origin is $\frac{x}{Lx'} = \frac{y}{My'} = \frac{z}{Nz'}$. If then the common value of these fractions be called ρ, we have $x' = \frac{x}{L\rho}$, $y' = \frac{y}{M\rho}$, $z' = \frac{z}{N\rho}$, substituting these values in $Lx'^2 + My'^2 + Nz'^2 = 0$, we get $\frac{x^2}{L} + \frac{y^2}{M} + \frac{z^2}{N} = 0$. The form of the equation shows that the relation between the cone is reciprocal, and that the edges of the first are perpen-

dicular to the tangent planes to the second. It can easily be seen that this is a particular case of the last example.

If the equation of the cone be given in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

the equation of the reciprocal cone will be the same as that of the reciprocal curve in plane geometry, viz.

$$(bc - f^2)x^2 + (ca - g^2)y^2 + (ab - h^2)z^2 + 2(gh - af)yz + 2(hf - bg)zx + 2(fg - ch)xy = 0.$$

Ex. 13*a*. If U denote the terms of highest degree in the equation, and S denote

$$(bc - f^2)x^2 + (ca - g^2)y^2 + (ab - h^2)z^2 + 2(gh - af)yz + 2(hf - bg)zx + 2(fg - ch)xy,$$

then the equation of the three principal planes, the centre being origin, is denoted by the determinant

$$\begin{vmatrix} x, & y, & z \\ U_1, & U_2, & U_3 \\ S_1, & S_2, & S_3 \end{vmatrix} = 0.$$

Ex. 14. A line moves about so that three fixed points on it move on fixed planes; to find the locus of any other point on it.

Let the coordinates of the locus point P be α, β, γ; and let the three fixed planes be taken for coordinate planes meeting the line in points A, B, C. Then it is easy to see that the coordinates of A are $0, \frac{AB}{PB}\beta, \frac{AC}{PC}\gamma$, where the ratios $AB . PB$, AC PC are known. Expressing then, by Art. 10, that the distance PA is constant, the locus is at once found to be an ellipsoid.

Ex. 15. A and O are two fixed points, the latter being on the surface of a sphere. Let the line joining any other point D on the sphere to A meet the sphere again in D'. Then if on OD a portion OP be taken $= AD'$, find the locus of P. [W. R. Hamilton.]

We have $AD^2 = AO^2 + OD^2 - 2AO . OD \cos AOD$. But AD varies inversely as the radius vector of the locus, and OD is given, by the equation of the sphere, in terms of the angles it makes with fixed axes. Thus the locus is easily seen to be a quadric of which O is the centre.

Ex. 16. A plane passes through a fixed line, and the lines in which it meets two fixed planes are joined by planes each to a fixed point; find the surface generated by the line of intersection of the latter two planes.

[Let the fixed line meet the fixed planes in A and B, and let C, D be the fixed points. The lines whose locus is sought are the lines of intersection of homographic pencils of planes through AC and BD, and the locus is therefore a quadric.]

Ex. 17. The locus of a point, the feet of the perpendiculars from which on the faces of the tetrahedron of reference lie in a plane, is

$$\frac{A}{x} + \frac{B}{y} + \frac{C}{z} + \frac{D}{w} = 0$$

x, y, z, w being actual perpendiculars, and A, B, C, D the areas of the face of the tetrahedron.

CHAPTER VII.

RECIPROCATION, DUALITY, ABRIDGED NOTATION, AND PROJECTION.

Reciprocation and the Principle of Duality.

122. We shall in this chapter give examples of the application to quadrics of methods of abridged notation. It is convenient, however, first to show that every figure we employ admits of a two-fold description, and that every theorem we obtain is accompanied by another reciprocal theorem. In fact, the reader can see without difficulty that the whole theory of Reciprocal Polars explained (*Conics*, Chap. XV.) is applicable to space of three dimensions. Being given a fixed quadric, S, and any surface U, we can generate a new surface V by taking the pole with regard to S of every tangent plane to U. If we have a point on V *corresponding* to a tangent plane of U, reciprocally the tangent plane to V at that point will correspond to the point of contact of the tangent plane to U. For the tangent plane to V contains all the points on V consecutive to the assumed point; and to it must correspond the point through which pass all the tangent planes of U consecutive to the assumed tangent plane; that is to say, the point of contact of that plane. Thus *to every point connected with one surface corresponds a plane connected with the other, and vice versâ; and to a line (joining two points) corresponds a line (the intersection of two planes)*. This is the Principle of Duality. For example the degree of V, being measured by the number of points in which an arbitrary line meets it, is equal to the number of tangent planes which can be drawn to U through an arbitrary right line. Thus the reciprocal of a quadric is a quadric, since two tangent planes

can be drawn to a quadric through any arbitrary right line (Art. 80).

[The reciprocal V of a surface U may be defined either as the locus of poles, with regard to S, of tangent planes to U, or as the envelope of polar planes, with regard to S, of points on U; and V is similarly related to U.]

[Ex. Find the reciprocals of the general quadric, and of the cubic surface $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w} = 0$, with regard to the quadric $x^2 + y^2 + z^2 + w^2 = 0$. Cf. Ex. 12, Art. 121.]

123. In order to show what corresponds to a curve in space we shall anticipate a little of the theory of curves of double curvature to be explained hereafter. A curve in space may be considered as a series of points in space, 1, 2, 3, &c., arranged according to a certain law. If each point be joined to its next consecutive point, we shall have a series of lines 12, 23, 34, &c., each line being a tangent to the given curve. The assemblage of these lines forms a surface, and a *developable* surface (see note, Art. 112), since any line 12 intersects the consecutive line 23. Again, if we consider the planes 123, 234, 345, &c., containing every three consecutive points, we shall have a series of planes which are called the *osculating* planes of the given curve, and which are tangent planes to the developable generated by its tangents.* Now when we reciprocate, it is plain that to the series of points, lines, and planes will correspond a series of planes, lines, and points; and thus, that *the reciprocal of a series of points forming a curve in space will be a series of planes touching a developable. If the curve in space lies all in one plane, the reciprocal planes will all pass through one point, and will be tangent planes to a cone.*

* [Strictly speaking a continuous curve has no consecutive points, since between any two points there lies another. In intuitional geometry it may be regarded as a polygon containing consecutive vertices. But the curve is not a polygon, but the limiting locus obtained by supposing that the sides of the polygon are smaller than any assignable length.]

The series of points common to two surfaces forms a curve. Reciprocally the series of tangent planes common to two surfaces touches a developable which envelopes both surfaces. To the series of tangent planes (enveloping a *cone*) which can be drawn to the one surface through any point, corresponds the series of points on the other which lie in the corresponding plane: that is to say, *to a plane section of one surface corresponds a tangent cone of the reciprocal.* It easily follows hence, that to a point and its polar plane with respect to a quadric, correspond a plane and its pole with respect to the reciprocal quadric.

[Ex. 1. Prove that the envelope of a plane $\lambda x + \mu y + \nu z + \rho w = 0$, whose coordinates λ, μ, ν, ρ, satisfy two equations, is a developable. Reciprocate with regard, say, to $x^2 + y^2 + z^2 + w^2 = 0$ and this is deduced from the known theorem that the locus of points whose coordinates satisfy two equations is a curve.

Ex. 2. If one of the equations connecting λ, μ, ν, ρ is linear the developable reduces to a cone.

Ex. 3. Find the condition that the planes satisfying

$$(a, b, c, d, f, g, h, l, m, n)(\lambda, \mu, \nu, \rho)^2 = 0$$

may be tangent planes to a conic, and find the point-equation of the plane of the conic. The reciprocal problems are to find the condition that the general quadric may represent a cone and to find the vertex of the cone.]

Reciprocation with regard to a sphere.

124. The reciprocals are frequently taken with regard to a sphere whose centre is called the *origin of reciprocation*, and as at *Conics* (Art. 307) mention of the sphere may be omitted, and the reciprocals spoken of as taken with regard to this origin. To the origin will evidently correspond the plane at infinity; and to the section of one surface by the plane at infinity will correspond the tangent cone which can be drawn to the other through the origin. Thus, then, when the origin is *without* a quadric, that is to say, is such that real tangent planes can be drawn from it to the surface,* the reciprocal surface will have real points at infinity, that is to say, will be a

* [This is always possible for a ruled quadric since the tangent planes are the planes joining the point to the generators.]

hyperboloid; when the origin is inside, the reciprocal is an ellipsoid; when the origin is *on* the surface, the reciprocal will be touched by the plane at infinity, or what is the same thing (Art. 116) the reciprocal is a paraboloid.

The reciprocal of a *ruled* surface (that is to say, of a surface generated by the motion of a right line) is a ruled surface. For to a right line corresponds a right line, and to the surface generated by the motion of one right line will correspond the surface generated by the motion of the reciprocal line. Hence to a hyperboloid of one sheet always corresponds a hyperboloid of one sheet unless the origin be on the surface when the reciprocal is a hyperbolic paraboloid.

Cayley has remarked, that *the degree of any ruled surface is equal to the degree of its reciprocal.* The degree of the reciprocal is equal to the number of tangent planes which can be drawn through an arbitrary right line. Now it will be formally proved hereafter, but is sufficiently evident in itself, that the tangent plane at any point on a ruled surface contains the generating line which passes through that point. The degree of the reciprocal is therefore equal to the number of generating lines which meet an arbitrary right line. But this is exactly the number of points in which the arbitrary line meets the surface, since every point on a generating line is a point on the surface.*

125. When reciprocals are taken with regard to a sphere, any plane is evidently perpendicular to the line joining the corresponding point to the origin. Thus to any cone corresponds a plane curve, and the cone whose base is that curve and vertex the origin has an edge perpendicular to every tangent plane of the first cone, and *vice versâ*. In general two cones (which may or may not have a common vertex) are said to be reciprocal when every edge of one is perpendicular to a tangent plane of the other (see Ex. 13, Art. 121). For example, it appears from the last article, that the tangent cone from the origin to any surface is in this sense reciprocal to the asymptotic cone of the reciprocal surface.

The sections by any plane of two reciprocal cones, having a common vertex, are polar reciprocals with regard to the

* [This theorem takes a special form when the ruled surface is a developable (see Note Art. 327).]

foot of the perpendicular on that plane from the common vertex. For, let the plane meet an edge of one cone in a point P, and the perpendicular tangent plane to the other in the line QR; let M be the foot of the perpendicular on the plane from the vertex O; then it is easy to see that the line PM is perpendicular to QR; and if it meet it in S, then since the triangle POS is right-angled, the rectangle $PM \cdot MS$ is equal to the constant OM^2. The curve therefore which is the locus of the point P is the same as that got by letting fall from M perpendiculars on the tangents QR, and taking on each perpendicular a portion inversely as its length.

The following illustrates the application of the principle here established: *Through the vertex of any cone of the second degree can be drawn two lines, called focal lines, such that the section of the cone by a plane perpendicular to either line is a conic, having for a focus the point where the plane meets the focal line.* For form a reciprocal cone by drawing through the vertex lines perpendicular to the tangent planes of the given cone; then this cone has two planes of circular section (Art. 104); and, by the present article, the section of the given cone by a plane parallel to either is a conic having for a focus the foot of the perpendicular on that plane from the vertex. [For the reciprocal of a circle is a conic, having the origin for its focus and the line corresponding to the centre of the circle for directrix.] What has been just proved may be stated, *the focal lines of a cone are perpendicular to the planes of circular section of the reciprocal cone.*

126. *The reciprocal of a sphere with regard to any point is a surface generated by the revolution of a conic round the transverse axis.* This may be proved as the corresponding theorem for the reciprocal of a circle (*Conics*, Art. 308). It is easily proved that if we have any two points A and B, the distances of these two points from the origin are in the same ratio as the perpendiculars from each on the plane corresponding to the other (*Conics*, Art. 101). Now the distance of the centre of a fixed sphere from the origin, and the perpendicular

from that centre on any tangent plane to the sphere are both constant. Hence, any point on the reciprocal surface is such that its distance from the origin is in a constant ratio to the perpendicular let fall from it on a fixed plane; namely, the plane corresponding to the centre of the sphere. And this locus is manifestly a surface of revolution, of which the origin is a focus; and the plane in question a directrix plane.

[The following principles should be noticed: (a) The angle between two planes is equal to the angle between the lines joining the corresponding points to the origin; (b) The line corresponding to a given line l is in the plane through the original perpendicular to l; (c) If the angle between a line and a plane P is a right angle, the angle made by the plane joining the corresponding line to O with the line joining O to the point corresponding to P is a right angle, and conversely; (d) If the angle between two lines is a right angle, the angle between the planes joining the corresponding lines to O is also a right angle.]

By reciprocating properties of the sphere we thus get properties of surfaces of revolution round the transverse axis. The left-hand column contains properties of the sphere, the right-hand those of the surfaces of revolution.

Ex. 1. Any tangent plane to a sphere is perpendicular to the line joining its point of contact to the centre.	The line joining the focus to any point on the surface is perpendicular to the plane through the focus and the intersection with the directrix plane of the tangent plane at the point.
Ex. 2. Every tangent cone to a sphere is a right cone, the tangent planes all making equal angles with the plane of contact.	The cone whose vertex is the focus and base any plane section is a right cone whose axis is the line joining the focus to the pole of the plane of section.

A particular case of Ex. 2 is "Every plane section of a paraboloid of revolution is projected orthogonally into a circle on the tangent plane at the vertex." [Take the focus at infinity and the cone becomes a right cylinder with axis parallel to axis of paraboloid.]

Ex. 3. Any plane is perpendicular to the line joining the centre to its pole.	The line joining any point to the focus is perpendicular to the plane joining the focus to the intersection with the directrix plane of the polar plane of the point.

Ex. 4. Any plane through the centre is perpendicular to the conjugate diameter.

Any plane through the focus is perpendicular to the line joining the focus to its pole.

Ex. 5. The cone whose base is any plane section of a sphere has circular sections parallel to the plane of section.

Any tangent cone has for its focal lines the lines joining the vertex of the cone of the two foci. (Cf. Art. 125.)

Ex. 6. Every cylinder enveloping a sphere is right.

Every section passing through the focus has this focus for a focus.

Ex. 7. Any two conjugate * right lines are mutually perpendicular.

Any two conjugate lines are such that the planes joining them to the focus are at right angles.

Ex. 8. Any quadric enveloping a sphere is a surface of revolution; and its asymptotic cone therefore is a right cone.

If a quadric envelop a surface of revolution, the cone enveloping the former, whose vertex is a focus of the latter, is a cone of revolution.

127. The equation of the reciprocal of a central surface with regard to any point (x', y', z') is found as at *Conics*, Art. 319. For the length of the perpendicular from any point on tangent plane is (see Art. 89)

$$p = \frac{k^2}{\rho} = \sqrt{(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma)} - (x'\cos\alpha + y'\cos\beta + z'\cos\gamma),$$

and, moving the origin to (x', y', z'), the reciprocal is thus

$$(xx' + yy' + zz' + k^2)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

Thus the reciprocal with regard to the centre is

$$a^2x^2 + b^2y^2 + c^2z^2 = k^4,$$

a quadric whose axes are the reciprocals of the axes of the given one.

We have given (Ex. 12, Art. 121) the method in general of finding the equation of the reciprocal of one quadric with regard to another. Thus the reciprocal with regard to the sphere $x^2 + y^2 + z^2 = k^2$, is found by substituting x, y, z, $-k^2$ for α, β, γ, δ in the tangential equation, Art. 79; or, more

* The polar planes with respect to a quadric of all the points of a line pass through a right line, which we call the conjugate line, or polar line (Art. 65).

symmetrically, *the tangential equation itself may be considered as the equation of the reciprocal with regard to* $x^2+y^2+z^2+w^2=0$; α, β, γ, δ being the coordinates.

The reciprocal of the reciprocal of a surface is the surface itself (Art. 122). Considering the quadric, if we actually form the equation of the reciprocal of the reciprocal $A\alpha^2+B\beta^2+$ &c., the new coefficient of x^2 is $BCD+2FMN-BN^2-CM^2-DF^2$, which, when we substitute their values for B, C, &c., will be found to be $a\Delta^2$. And Δ^2 will in like manner be a factor in every term, so that the reciprocal of the reciprocal is the given equation multiplied by the square of the discriminant (see *Higher Algebra*, Art. 33, or Burnside and Panton, Art. 146).

Tangential Coordinates.

128. The principle of duality may be established independently of the method of reciprocal polars, by showing in extension of the remarks made above, Art. 38 (see *Conics*, Art. 299) that all the equations we employ admit of a two-fold interpretation; and that when interpreted as equations in tangential coordinates, they yield theorems reciprocal to those which they give according to the mode of interpretation hitherto adopted. We may call α, β, γ, δ the *tangential coordinates of the plane* $\alpha x+\beta y+\gamma z+\delta w$. Now the condition that this plane may pass through a given point, being

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0,$$

conversely, any equation of the first degree in α, β, γ, δ,

$$A\alpha+B\beta+C\gamma+D\delta=0$$

is the condition that this plane may pass through a point whose coordinates are proportional to A, B, C, D; and the equation just written may be regarded as *the tangential equation of that point.* If the tangential coordinates of two planes are α, β, γ, δ; α', β', γ', δ', it follows, from Art. 37, that $\alpha+k\alpha'$, $\beta+k\beta'$, &c. are the coordinates of a plane passing through the line of intersection of the two given planes. And again, it follows from Art. 8, that if $L=0$, $M=0$ be the tangential equations of two points, $L+kM=0$ denotes a point on the line joining the two given ones; and similarly

(Art. 9), that $L+kM+k'N$ denotes a point in the plane determined by the three points L, M, N.

Again, any equation in α, β, γ, δ may be considered as *the tangential equation of a surface* touched by every plane $\alpha x+\beta y+\gamma z+\delta w$ whose coordinates satisfy the given equation. If the equation be of the n^{th} order, the surface will be of the n^{th} class, or such that n tangent planes (fulfilling the given relation) can be drawn through any line. For if we substitute in the given equation $\alpha'+k\alpha''$, $\beta'+k\beta''$, &c. for α, β, &c., we get an equation of the n^{th} degree in k, determining n planes satisfying the given relation, which can be drawn through the intersection of the planes $\alpha'\beta'\gamma'\delta'$, $\alpha''\beta''\gamma''\delta''$.

129. The general tangential equation of the second degree

$$A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+2F\beta\gamma+2G\gamma\alpha+2H\alpha\beta$$
$$+2L\alpha\delta+2M\beta\delta+2N\gamma\delta=0$$

can be discussed by precisely the same methods as are used above (Arts. 75-80).* If we substitute $\alpha'+k\alpha''$, &c. for α, &c., we get a quadratic in k, which may be written

$$\Sigma'+2k\Pi+k^2\Sigma''=0.$$

If the plane $\alpha'\beta'\gamma'\delta'$ touch the surface in question, $\Sigma'=0$, and one of the roots of the quadratic is $k=0$. The second root will be also $k=0$, provided that $\Pi=0$. In other words, the coordinates of any tangent plane consecutive to $\alpha'\beta'\gamma'\delta'$ must satisfy the condition

$$\alpha\frac{d\Sigma'}{d\alpha'}+\beta\frac{d\Sigma'}{d\beta'}+\gamma\frac{d\Sigma'}{d\gamma'}+\delta\frac{d\Sigma'}{d\delta'}=0.$$

But this equation being of the first degree *represents a point, viz. the point of contact of* $\alpha'\beta'\gamma'\delta'$, through which every consecutive tangent plane must pass.

We may regard the relation just obtained as one connecting the coordinates of a tangent plane with those of any plane passing through its point of contact, and from the symmetry of this relation, we infer (as in Art. 63) that if α', β', γ', δ', be the coordinates of any plane, those of the tangent plane at every point of the surface which lies in that

* Cf. Art. 80*b*.

plane, must fulfil the condition

$$\alpha\frac{d\Sigma'}{d\alpha'}+\beta\frac{d\Sigma'}{d\beta'}+\gamma\frac{d\Sigma'}{d\gamma'}+\delta\frac{d\Sigma'}{d\delta'}=0.$$

But this equation represents a point through which all the tangent planes in question must pass; in other words, it *represents the pole of the given plane.*

We can, by following the process pursued in Art. 79, deduce from the general tangential equation of the second degree the corresponding equation to be satisfied by its points. If the tangential equation of any point on the surface be

$$x'\alpha+y'\beta+z'\gamma+w'\delta=0,$$

and $\alpha\beta\gamma\delta$ the coordinates of the corresponding tangent plane, we infer from the equations already obtained, that if λ be an indeterminate multiplier, we must have

$$\lambda x'=A\alpha+H\beta+G\gamma+L\delta,\quad \lambda y'=H\alpha+B\beta+F\gamma+M\delta,$$
$$\lambda z'=G\alpha+F\beta+C\gamma+N\delta,\quad \lambda w'=L\alpha+M\beta+N\gamma+D\delta.$$

Solving these equations for $\alpha\beta\gamma\delta$, we get the coordinates of the polar plane of any assumed point; and expressing that these coordinates satisfy the given tangential equation, we get the relation to be satisfied by the x, y, z, w of any point on the surface, a relation only differing by the substitution of capital for small letters from that found in Art. 79.

[The degree and class of a quadric are the same, or the quadric reciprocates into a quadric. The pure geometrical expression for this is to be found in the fact that a hyperboloid may be defined either as the locus of points on a line meeting three fixed lines, or as the envelope of planes through a line lying in a plane with (i.e. intersecting) three fixed lines. The property that every tangent plane contains two generators passing through the point of contact is equivalent to the reciprocal property that through every point two generators can be drawn which lie in the tangent plane. A generator is equally definable as a line such that all points on it lie on the surface, and as a line such that all planes through it are tangent planes (cf. Arts. 80*a*, 80*b*). A generator reciprocates into a generator.]

Abridged Notation.*

130. In what follows, we have only to suppose that the abbreviations denote equations in tangential coordinates,

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* [The theorems proved under this heading are *projective*, that is they are independent of linear transformation of coordinates. *Metrical* properties

when we get direct proofs of the reciprocals of the theorems actually obtained.

If U and V represent any two quadrics, then $U+\lambda V$ represents a quadric passing through *every* point common to U and V, and if λ be indeterminate it represents a series of quadrics having a common curve of intersection. Since nine points determine a quadric (Art. 58), $U+\lambda V$ is the most general equation of the quadric passing through eight given points (see *Higher Plane Curves*, Art. 29).* For if U and V be two quadrics, each passing through the eight points, $U+\lambda V$ represents a quadric also passing through the eight points, and the constant λ can be so determined that the surface shall pass through any ninth point, and can in this way be made to coincide with any given quadric through the eight points. It follows then that *all quadrics which pass through eight points have besides a whole series of common points, forming a common curve of intersection;* and reciprocally, that *all quadrics which touch eight given planes have a whole series of common tangent planes determining a fixed developable which envelopes the whole series of surfaces touching the eight fixed planes.*

It is evident also that the problem to describe a quadric through nine points may become indeterminate. For if the ninth point lie anywhere on the curve which, as we have just seen, is determined by the eight fixed points, then *every* quadric passing through the eight fixed points will pass through the ninth point, and it is necessary that we should be given a ninth point, *not* on this curve, in order to be able to determine the surface. Thus if U and V be two quadrics through the eight points, we determine the surface by substituting the coordinates of the ninth point in $U+\lambda V=0$;

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involving distances and magnitude of angles are not involved, since they are altered by linear transformation. Projection will be more fully considered in Art. 144 (c).]

* [If we express the conditions that a quadric may pass through eight given points, we can eliminate all the constants except one and λ may be taken for the remaining constant.]

but if these coordinates make $U=0$, $V=0$, this substitution does not enable us to determine λ.

131. *Given seven points (or tangent planes) common to a series of quadrics, then an eighth point (or tangent plane) common to the whole system is determined.*

For let U, V, W be three quadrics, each of which passes through the seven points, then $U+\lambda V+\mu W$ may represent *any* quadric which passes through them; for the constants λ, μ may be so determined that the surface shall pass through any two other points, and may in this way be made to coincide with any given quadric through the seven points. But $U+\lambda V+\mu W$ represents a surface passing through *all* points common to U, V, W, and since these intersect in eight points, it follows that there is a point, in addition to the seven given, which is common to the whole system of surfaces.

We see thus, that though it was proved in the last article that eight points *in general* determine a curve of double curvature common to a system of quadrics, it is *possible* that they may not. For we have just seen that there is a particular case in which to be given eight points is only equivalent to being given seven. When we say therefore that a quadric is determined by nine points, and that the intersection of two quadrics is determined by eight points, it is assumed that the nine or eight points are perfectly unrestricted in position.*

132. If a system of quadrics have a common curve of intersection, the polar plane of any fixed point passes through a fixed right line.

If a system of quadrics be inscribed in the same developable, the locus of the pole of a fixed plane is a right line.

* The reader who has studied *Higher Plane Curves*, Arts. 29-34, will have no difficulty in developing the corresponding theory for surfaces of any degree. Thus if we are given one less than the number of points necessary to determine a surface of the n^{th} degree, we are given a series of points forming a curve through which the surface must pass; and if we are given two less than the number of points necessary to determine the surface, then we are given a certain number of other points (namely as many as will make the entire number up to n^3) through which the surface must also pass.

For if P and Q be the polar planes of a fixed point with regard to U and V respectively, then $P+\lambda Q$ is the polar of the same point with respect to $U+\lambda V$.

In particular, the locus of the centres of all quadrics inscribed in the same developable is a right line.

133. *If a system of quadrics have a common curve of intersection (or be inscribed in a common developable), the polars of a fixed line generate a hyperboloid of one sheet.*

Let the polars of two points in the line be $P+\lambda Q$, $P'+\lambda Q'$, then it is evident that their intersection lies on the hyperboloid $PQ'=QP'$.

134. *If a system of quadrics have a common curve, the locus of the pole of a fixed plane is a curve in space of the third degree.* For, eliminating λ between $P+\lambda Q$, $P'+\lambda Q'$, $P''+\lambda Q''$, the polars of any three points, each determinant of the system

$$\left\| \begin{matrix} P, & P', & P'' \\ Q, & Q', & Q'' \end{matrix} \right\|$$

vanishes. Now the intersection of the surfaces represented by $PQ'=QP'$, $PQ''=QP''$, is a curve of the fourth degree, but this includes the right line PQ, which is not part of the intersection of $PQ''=QP''$, $P'Q''=Q'P''$. There is therefore only a curve of the third degree common to all three.

Reciprocally, *if a system be inscribed in the same developable, the polar of a fixed point envelopes the developable which is the reciprocal of a curve of the third degree*, being (as will afterwards be shown) a developable of the fourth order.

135. Given seven points on a quadric, the polar plane of a fixed point passes through a fixed point.

Given seven tangent planes to a quadric, the pole of a fixed plane moves in a fixed plane.

For evidently the polar of a fixed point with regard to $U+\lambda V+\mu W$ will be of the form $P+\lambda Q+\mu R$, and will therefore pass through a fixed point.*

* Hesse has derived from this theorem a construction for the quadric passing through nine given points. *Crelle*, Vol. XXIV. p. 36. *Cambridge and*

136. Since the discriminant contains the coefficients in the fourth degree, it follows that we have a biquadratic equation to solve to determine λ, in order that $U+\lambda V$ may represent a cone, and therefore that *through the intersection of two quadrics four cones may be described*. The vertex of each of these cones is the common intersection of the four planes,

$$U_1+\lambda V_1,\ U_2+\lambda V_2,\ U_3+\lambda V_3,\ U_4+\lambda V_4,$$

when λ satisfies the biquadratic just referred to, and the four vertices are got by substituting its four roots in succession in any three of these equations; they are therefore the four points common to the surfaces found by making each of the determinants

$$\left\| \begin{matrix} U_1, & U_2, & U_3, & U_4 \\ V_1, & V_2, & V_3, & V_4 \end{matrix} \right\| = 0.$$

There are *four points whose polars are the same with respect to all quadrics passing through a common curve of intersection*, namely the vertices of the four cones just referred to. For to express the conditions that

$$xU_1'+yU_2'+zU_3'+wU_4'=0,$$
$$xV_1'+yV_2'+zV_3'+wV_4'=0,$$

should represent the same plane, we find the very same set of determinants. In like manner there are *four planes whose poles are the same with respect to a set of quadrics inscribed in the same developable.*

[*The vertex of any cone of the system $U+\lambda V$ lies on the polar plane of any other vertex, with regard to any quadric of the system.* For if (x', y', z', w'), (x'', y'', z'', w'') are two vertices, then $\frac{U_1'}{V_1'}=\frac{U_2'}{V_2'}=\frac{U_3'}{V_3'}=\frac{U_4'}{V_4'}=-\lambda'$ where λ' is the corresponding root of the biquadratic. Each of these fractions

$$=\frac{x''U_1'+y''U_2'+z''U_3'+w''U_4'}{x''V_1'+y''V_2'+z''V_3'+w''V_4'},$$

which by similar reasoning $=-\lambda''$, if the numerator and denominator are different from zero. But the roots of the biquadratic are not in general equal; hence the numerator and denominator are both zero and therefore the polar of each point passes through the other. Thus the four points considered are *the vertices of a tetrahedron such that each face is the polar plane of the opposite vertex with regard to any quadric of the system $U+\lambda V$.* The

Dublin Mathematical Journal, Vol. IV. p. 44. See also some further developments of the same problem by Townsend, *ib.* Vol. IV. p. 241.

tetrahedron thus uniquely determined by U and V is called the *self-conjugate* or *self-polar tetrahedron* of the system $U + \lambda V$.

The self-conjugate tetrahedron reciprocates into a self-conjugate tetrahedron, and thus the system $U + \lambda V$ and the system $\Sigma + \mu\Sigma'$ have the same self-conjugate tetrahedron, Σ, Σ' being the tangential equations of U and V.

By reciprocating we see that there are *four values of μ for which $\Sigma + \mu\Sigma'$ represents a conic, and these four conics lie on the faces of the self-conjugate tetrahedron.* They are touched by the tangent planes to the developable circumscribing the system.]

137. If the surface V break up into two planes, the form $U+\lambda V=0$, becomes $U+\lambda LM=0$, a case deserving of separate examination.* In general the intersection of two quadrics is a curve of double curvature of the fourth degree, which may in some cases (Art. 134) break up into a right line and a cubic, but the intersection with U of any of the surfaces $U+\lambda LM$, evidently reduces to the two conics in which U is cut by the planes L and M. *Any point on the line LM has the same polar plane with regard to all surfaces of the system $U+\lambda LM$.*† For if P be the polar of any point with regard to U, its polar with regard to $U+\lambda LM$ will be

$$P+\lambda\,(LM'+ML')$$

which reduces to P, when $L'=0$, $M'=0$. Thus, in particular, at each of the two points where the line LM meets U, all the surfaces have the same tangent plane. The form, then, $U+\lambda LM$, may be regarded as denoting a system of quadrics having double contact with each other.

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* The case where U also breaks up into two planes has been discussed, Art. 108.

† There are two other points whose polar planes are the same with regard to all the quadrics, and which therefore (Art. 136) will be vertices of cones containing both the curves of section. It is only necessary that P, the polar plane of one of these points with regard to U, should be the same plane as $ML' + LM'$ the polar with regard to LM. Since then the polar plane of the point with regard to U passes through LM, the point itself must lie on the polar line of LM with regard to U, that is to say, on the intersection of the tangent planes where LM meets U. Let this polar line meet U in AA', and LM in BB', then the points required will be FF', the foci of the involution determined by AA', BB'. For since FF' form a harmonic system either with AA' or with BB', the polar plane of F either with regard to U or LM passes through F', and *vice versâ*.

Conversely, *if two quadrics have double contact, their curve of intersection breaks up into simpler curves.* For if we draw any plane through the two points of contact and through any point of their intersection, this plane will meet the quadrics in sections having three points common, and having common also the two tangents at the points of contact; these sections must therefore be identical, and the curve of intersection breaks up into two plane curves; unless the line joining the points of contact be a generator of each surface, in which case the rest of the curve of intersection is a curve of the third degree.

[From the preceding it follows that the general type of a family of quadrics having double contact with each other at the points where they meet the line of intersection of two fixed planes LM, all the quadrics having the same tangent plane at these points, is $U + \lambda L^2 + 2\mu LM + \nu M^2 = 0$; the family is therefore triply infinite, depending on three arbitrary constants λ, μ, ν. The reciprocal form is $\Sigma + p\alpha^2 + 2q\alpha\beta + r\beta^2$.]

Reciprocally, if two quadrics have double contact they are enveloped by two cones of the second degree. For take the point where the intersection of the two given common tangent planes is cut by any other common tangent plane; then the cones having this point for vertex, and enveloping each surface, have common three tangent planes and two lines of contact, and are therefore identical. The reciprocals of a pair of quadrics having double contact will manifestly be a pair of quadrics having double contact, and the two planes of intersection of the one pair will correspond to the vertices of common tangent cones to the other pair.

138. *If there be a plane curve common to three quadrics, each pair must have also another common plane curve, and the three planes of these last common curves pass through the same line.* Let the quadrics be U, $U + LM$, $U + LN$, then the last two have evidently for their mutual intersection two plane sections made by L, $M - N$.

Ex. Reciprocate this theorem.

139. Similar quadrics belong to the class now under dis-

cussion. Two quadrics are similar and similarly placed when the terms of the second degree are the same in both (see *Conics*, Art. 234). Their equations then are of the form $U=0$, $U+cL=0$. We see then that *two similar and similarly situated quadrics intersect in general in one plane curve, the other plane of intersection being at infinity* and common to all quadrics of the system $U+cL=0$. If there be three quadrics, similar and similarly placed, their three finite planes of intersection pass through the same right line.

Spheres are all similar and similarly situated quadrics, and therefore are to be considered as having a common section at infinity, namely an imaginary circle.

A plane section of a quadric will be a circle if it passes through the two points in which its plane meets this imaginary circle at infinity. We may see thus immediately of how many solutions the problem of finding *the circular sections of a quadric* is susceptible. For the section of the quadric by the plane at infinity meets the section of a sphere by the same plane in four points, which can be joined by *six* right lines, the planes passing through any one of which meet the quadric in a circle. The six right lines may be divided into three pairs, each pair intersecting in one of the three points whose polars are the same with respect to the section of the quadric and of the sphere. And we shall see that these three points determine the *directions of the axes of the quadric.*

An umbilic (Art. 106) is the point of contact of a tangent plane which can be drawn through one of these six right lines. There are in all therefore *twelve umbilics, though only four are real.* If a tangent plane be drawn to a quadric through any line, the generators in that tangent plane evidently pass, one through each of the points where the line meets the surface. Thus, then, the umbilics must lie each on some one of the eight generators, which can be drawn through the four points at infinity common to the quadric and any sphere. Or, as Hamilton has remarked, the *twelve umbilics lie three by three on eight imaginary right lines.*

A surface of revolution is one which has double contact

at infinity with a sphere. For an equation of the form $x^2+y^2+az^2=b$ can be written in the form

$$(x^2+y^2+z^2-r^2)+\{(a-1)\,z^2-(b-r^2)\}=0,$$

and the latter part represents two planes. It is easy to see then why in this case there is but one direction of real circular sections, determined by the line joining the points of contact of the sections at infinity of a sphere and of the quadric.

[This paragraph contains the *projective* definition of similar quadrics, spheres, surfaces of revolution, &c. (see note Art. 130). Hence we can infer projective relations involving such surfaces. The relation of *rectangularity* can also be defined projectively. For two perpendicular lines are conjugate directions with regard to a sphere; they are therefore lines meeting the plane at infinity in conjugate points with regard to the imaginary circle at infinity. Similarly a line is perpendicular to a plane when their intersection with the plane at infinity are pole and polar with regard to the circle; and two perpendicular planes meet the plane at infinity in conjugate lines with regard to the same circle. Hence since *the axes of a quadric* are conjugate with regard to the quadric, and being perpendicular are also conjugate directions with regard to any sphere, they pass through the vertices of the common self-conjugate triangle of the imaginary circle at infinity and the section of the quadric by the plane at infinity.]

140. If the two planes L, M coincide, the form $U+\lambda LM$ becomes $U+\lambda L^2$, which denotes *a system of surfaces touching U at every point of the section of U by the plane L.* Two quadrics cannot touch in three points without their touching all along a plane curve (unless they have two common generators). For the sections by the plane of the three points and the cones which touch along them would be identical. The equation of the tangent cone to a quadric given, Art. 78, is a particular case of the form $U=L^2$. Also two concentric and similar quadrics (U, $U-c^2$) are to be regarded as having plane contact with each other, the plane of contact being at infinity. Any plane obviously cuts the surfaces U and $U-L^2$ in two conics having double contact with each other, and if the section of one reduce to a point-circle, that point must plainly be the focus of the other. Hence *when one quadric has plane contact with another, the tangent plane at the umbilic of one cuts the other in a conic*

of which the umbilic is the focus; and if one surface be a sphere, *every* tangent plane to the sphere meets the other surface in a section of which the point of contact is the focus.

Or these things may be seen by taking the origin at the umbilic and the tangent plane for the plane of xy, when on making $z=0$, the quantity $U - L^2$ reduces to $x^2 + y^2 - l^2$, and denotes a conic of which the origin is the focus, and l the directrix.

Two quadrics having plane contact with the same third quadric intersect each other in plane curves. Obviously $U - L^2$, $U - M^2$, have the planes $L - M$, $L + M$ for their planes of intersection.

141. The equation $aL^2 + bM^2 + cN^2 + dP^2$, where L, M, N, P represent planes, denotes a quadric such that any one of these four planes is the polar of the intersection of the other three. For $aL^2 + bM^2 + cN^2$ denotes a cone having the point LMN for its vertex; and the equation of the quadric shows that this cone touches the quadric, P being the plane of contact. The four planes form what we have called a *self-conjugate* tetrahedron with regard to the surface. It has been proved that, if two quadrics do not touch, there exist four planes forming a common self-conjugate tetrahedron. If these be taken for the planes L, M, N, P, the equations of both can be transformed to the forms

$$aL^2 + bM^2 + cN^2 + dP^2 = 0,\ a'L^2 + b'M^2 + c'N^2 + d'P^2 = 0.$$

It may also be seen, *à priori*, that this is a form to which it must be possible to bring the system of equations of two quadrics. For L, M, N, P involve implicitly three constants each; and the equations written above involve explicitly three independent constants each. The system therefore includes eighteen constants, and is therefore sufficiently general to express the equations of any two quadrics.*

* We are misled, however, if we conclude in like manner that the equations of any three quadrics may be written in the form

$$aL^2 + bM^2 + cN^2 + dP^2 + eQ^2 = 0,$$
$$a'L^2 + b'M^2 + c'N^2 + d'P^2 + e'Q^2 = 0,$$
$$a''L^2 + b''M^2 + c''N^2 + d''P^2 + e''Q^2 = 0,$$

142. *The lines joining the vertices of any tetrahedron to the corresponding vertices of its polar tetrahedron with regard to a quadric belong to the same system of generators of a hyperboloid of one sheet, and the intersections of corresponding faces of the two tetrahedra possess the same property.*

Taking the fundamental tetrahedron and its polar, the vertices of the polar tetrahedron (Art. 79) are proportional to the horizontal rows in

$$\begin{matrix} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & N, \\ L, & M, & N, & D. \end{matrix}$$

Thus the equations of the four lines we are considering are

$$\frac{y}{H}=\frac{z}{G}=\frac{w}{L},\ \frac{z}{F}=\frac{w}{M}=\frac{x}{H},$$

$$\frac{w}{N}=\frac{x}{G}=\frac{y}{F},\ \frac{x}{L}=\frac{y}{M}=\frac{z}{N}.$$

Now the condition that any line

$$\alpha x+\beta y+\gamma z+\delta w=0,\ \alpha' x+\beta' y+\gamma' z+\delta' w=0,$$

should intersect the first of the four, is, by eliminating x between the last two equations, found to be

$$H(\alpha\beta'-\beta\alpha')+G(\alpha\gamma'-\gamma\alpha')+L(\alpha\delta'-\delta\alpha')=0,$$

and the conditions that it should intersect each of the other three, are in like manner found to be

$$H(\beta\alpha'-\beta'\alpha)+F(\beta\gamma'-\beta'\gamma)+M(\beta\delta'-\beta'\delta)=0,$$

$$G(\gamma\alpha'-\gamma'\alpha)+\ F(\gamma\beta'-\gamma'\beta)+\ N(\gamma\delta'-\gamma'\delta)=0,$$

$$L(\delta\alpha'-\delta'\alpha)+M(\delta\beta'-\delta'\beta)+\ N(\delta\gamma'-\delta'\gamma)=0.$$

But these four conditions added together vanish identically. Any right line therefore which intersects the first three will

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where L, M, N, P, Q are five planes whose equations are connected by the relation

$$L+M+N+P+Q=0.$$

For though, since L, M, N, P, Q involve implicitly three constants each and the equations written above involve explicitly four independent constants each, the system thus appears to include twenty-seven constants, it has not really so many. For, as we shall show in a subsequent chapter, a relation must subsist among them, and the system is consequently not general enough to express the equations of any three quadrics.

intersect the fourth, which is, in other words, the thing to be proved.*

We find the equation of the hyperboloid by any of the methods in Art. 113, for example, by expressing that the line $\frac{wx'-w'x}{s}=\frac{wy'-w'y}{t}=\frac{wz'-w'z}{u}$ meets the first three of these lines. For then

$$\frac{Hw-Ly}{t}=\frac{Gw-Lz}{u},\quad \frac{Fw-Mz}{u}=\frac{Hw-Mx}{s},$$

$$\frac{Gw-Nx}{s}=\frac{Fw-Ny}{t},$$

from which by multiplication, s, t, u are eliminated in the form

$$(Fw-Mz)\,(Gw-Nx)\,(Hw-Ly)$$
$$=(Fw-Ny)\,(Gw-Lz)\,(Hw-Mx),$$

or $(HN-GM)\,(Fwx+Lyz)+(FL-HN)\,(Gwy+Mzx)$
$$+(GM-FL)\,(Hwz+Nxy)=0.$$

142*a*. This hyperboloidal relation between the four joining lines has been established by M'Cay by the following considerations.

First, considering any solid angle formed by three planes; their poles in regard to any quadric determine a plane, and in it these three poles form a triangle which is conjugate, in regard to the curve of section, to the triangle which the solid angle cuts out in the same plane.

Now conjugate triangles are in perspective, hence the three planes,—each through an edge of the solid angle, and the pole of its opposite face,—all pass through a right line.

If then we have two tetrahedra, polars with regard to a quadric, having the vertices $abcd$, $a'b'c'd'$, we see that at any one (a) of their eight vertices a right line may be found in the manner described; and since this line is common to the three planes abb', acc', add', it meets the connecting lines bb', cc', dd'; also, since it passes through (a) it meets aa'. In this way, taking each of the eight vertices, we have eight lines

* This theorem is due to Chasles. The proof here given is by Ferrers, *Quarterly Journal of Mathematics* (Vol. 1. p. 241).

each of which meets aa', bb', cc', dd'. The relation is thus demonstrated.

N.B. It appears from what has been stated that, when three planes are given and two points assumed which are to be poles to two of them in regard to any quadric, the locus of the pole of the third is a plane.

Ex. 1. Given three planes and their poles in regard to a quadric, the locus of the pole of a given plane is a right line (M'Cay). [The points and planes cannot be chosen arbitrarily, for the triangle formed by the points and the triangle formed by the lines in which the planes meet the plane of the points, must be in perspective. The locus is then a right line.]

[Ex. 2. All quadrics of the system mentioned in Ex. 1 have in common the plane section in which they meet the plane containing the three fixed points; and they have the same tangent plane at every point of this section.]

Ex. 3. The four perpendiculars from the vertices on the opposite faces in any tetrahedron are generators of one system, and the four perpendiculars to the faces at their orthocentres are generators of the other system of an equilateral hyperboloid.*

In the tetrahedron, whose vertices are a, b, c, d, let the opposite faces be A, B, C, D, and the perpendicular from a on A, x_0, from b on B, y_0, &c. Also let the feet of these perpendiculars be α, β, γ, δ. Then since in a spherical triangle the perpendiculars intersect, the planes through each edge of the solid angle (a) perpendicular to the opposite face intersect in a right line. This right line, therefore, meets the perpendiculars y_0, z_0, w_0, and as it passes through (a) it also meets x_0. In like manner at each other vertex we have a right line meeting those four right lines. They, therefore, belong to the same system of generators of a hyperboloid.

Again, taking through y_0 a parallel plane ϵ to x_0, this plane is orthogonal both to B and also to A, and, therefore, to their edge of intersection cd. Therefore this plane passes through a perpendicular of the triangle A, from vertex b on side cd.

Repeating this we see that the plane ϵ' through z_0 parallel to x_0 passes through the perpendicular from c on bd in the same triangle A. Thus the intersection $\epsilon\epsilon'$, which is parallel to x_0, is the perpendicular to A at its orthocentre. This line $\epsilon\epsilon'$ meets y_0, z_0, and x_0 (the latter at infinity), therefore it is a generator of the second system of the above hyperboloid, which contains the four perpendiculars of the tetrahedron.

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* The equilateral hyperboloid is defined as one which admits of three generators mutually at right angles, see Ex. 8, Art. 121. Schröter, as there referred to p. 205, gives these theorems. The first part of the theorem was given by Steiner, *Crelle* 2, p. 98. The second part of the theorem and the determination of the centre (Ex. 3) are referred by Baltzer to Joachimsthal, *Grunert Archiv*, 32, p. 109. Ex. 4 is referred to Monge, *Corresp. sur l'Ecole Polytech.* II. p. 266.

Further, the plane A intersects this hyperboloid in a conic, which passes through bcd and the orthocentre of A, which is, therefore, an equilateral hyperbola; the generators parallel to the asymptotes of this hyperbola and the generator x_0 are an orthogonal system, therefore the hyperboloid is equilateral.

This example is included in the general theorem [by using the projective definition of perpendicularity (Art. 139)].

Ex. 4. If in a tetrahedron a plane be taken through the middle of each edge normal to the opposite edge, these six planes intersect in a point, the centre of the above equilateral hyperboloid.

Ex. 5. In a tetrahedron the line joining the centre of the circumscribed sphere and the centre of the above equilateral hyperboloid is bisected by the centre of gravity of the tetrahedron.

143. The second part of the theorem stated in Article 142 is only the polar reciprocal of the first, but, as an exercise, we give a separate proof of it.

Taking the fundamental tetrahedron and its polar as before, the equations of the four lines are

$$\begin{aligned} x=0,\ hy+\ gz+\ lw&=0,\\ y=0,\ hx+\ fz+mw&=0,\\ z=0,\ gx+\ fy+\ nw&=0,\\ w=0,\ lx+my+\ nz&=0. \end{aligned}$$

Now the conditions that any line

$$ax+\beta y+\gamma z+\delta w=0,\ a'x+\beta' y+\gamma' z+\delta' w=0,$$

should intersect each of these are found to be (Art. 53b)

$$\begin{aligned} hv-g\tau+l\pi=0,\quad -hv+f\sigma+m\kappa=0,\\ g\tau-f\sigma+n\rho=0,\quad -l\pi-m\kappa-n\rho=0, \end{aligned}$$

and, as before, the theorem is proved by the fact that these conditions when added vanish identically. The equation of the hyperboloid is found to be

$$\begin{aligned} x^2ghl+y^2hfm+z^2fgn+w^2lmn\\ +(fyz+lxw)\ (gm+hn)+(gzx+myw)\ (hn+fl)\\ +(hxy+nzw)\ (fl+gm)=0. \end{aligned}$$

As a particular case of these theorems the lines joining each vertex of a circumscribing tetrahedron to the point of contact of the opposite face are generators of the same hyperboloid.

144. *Pascal's theorem* for conics may be stated as follows: "The sides of any triangle intersect a conic in six points

lying in pairs on three lines which intersect each the opposite side of the triangle in three points lying in one right line." Chasles has stated the following as an *analogous theorem for space of three dimensions:* "The edges of a tetrahedron intersect a quadric in twelve points, through which can be drawn four planes, each containing three points lying on edges passing through the same angle of the tetrahedron; then the lines of intersection of each such plane with the opposite face of the tetrahedron are generators of the same system of a certain hyperboloid."

Let the faces of the tetrahedron be x, y, z, w, and the quadric

$$x^2+y^2+z^2+w^2-\left(f+\frac{1}{f}\right)yz-\left(g+\frac{1}{g}\right)zx-\left(h+\frac{1}{h}\right)xy$$
$$-\left(l+\frac{1}{l}\right)xw-\left(m+\frac{1}{m}\right)yw-\left(n+\frac{1}{n}\right)zw,$$

then the four planes may be written

$$x=hy+gz+lw,\quad y=hx+fz+mw,$$
$$z=gx+fy+nw,\quad w=lx+my+nz,$$

whose intersections with the planes x, y, z, w, respectively, are a system of lines proved in the last article to be generators of the same hyperboloid.

144*a*. The conception of a *Brianchon's hexagon* may be extended to space, and we may denote by this name *any hexagon whose diagonals meet in a point.* Now it is evident that if this be the case, each pair of opposite sides of the hexagon intersect; and, conversely, if in any skew hexagon each pair of opposite sides intersect, the diagonals are concurrent. Thus three alternate sides of such a hexagon are met each by the other three, hence the odd sides belong to one set of generators of a hyperboloid of one sheet and the even to the other. Conversely, any hexagon whose sides lie in a hyperboloid is a Brianchon's hexagon.*

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* See a posthumous paper of O. Hesse in the 85th vol. of the Journal founded by Crelle; where, after giving the algebraical treatment of the above geometrically evident statements, Hesse also treats algebraically the question

It is further not difficult to see that if any hexagon U in space and a point (a) are given, and through (a) three right lines are drawn cutting the opposite sides of the hexagon in pairs, their intersections on consecutive sides of U are consecutive vertices of a Brianchon's hexagon V, having (a) as its Brianchon point. This hexagon V inscribed in U determines uniquely a hyperboloid on which it lies. But again this hyperboloid is cut by the sides of the given hexagon U in six other points, which in the same order are the vertices of a second Brianchon's hexagon inscribed in the given one and lying on the same hyperboloid, but having a different Brianchon point.

144*b*. Considering further this conception of a Brianchon's hexagon, there is at each vertex a tangent plane, and this contains the two sides which meet in that vertex. Now, taking an opposite pair of these six planes, viz. the plane containing the lines 1, 2, and the plane containing the lines 4, 5; since 1 meets 4 and 2 meets 5, the line of intersection of these two tangent planes is the same as the line joining the point 1, 4 to 2, 5. In like manner, the axis of 2, 3 with 5, 6 is the same as the ray from 2, 5 to 3, 6; and the axis of 3, 4 with 6, 1 is the same as the ray from 3, 6 to 1, 4. Hence, *the three axes of intersection of opposite tangent planes at the six points are coplanar.* Their plane may be considered a *Pascal plane* to the same hexagon. Thus, in three dimensions both properties meet in the same figure. [Each property is the reciprocal of the other; a hexagon on a quadric reciprocates into a hexagon on the reciprocal quadric, a generator being definable either as a line lying on the surface or as a line such that all planes through it are tangent planes (cf. Art. 129).] In fact—

If the surface and the tangent planes be *cut by an arbitrary plane* (A), since each tangent plane contains two generators, it will meet (A) in the chord joining two points

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of the two inscribed Brianchon's hexagons derived by aid of an arbitrary point from any skew hexagon.

on the conic of section, and what we have called the Pascal plane will meet (A) in the Pascal line of the inscribed hexagon.

But if the whole figure *be looked at from any point* (a) each generator of the surface determines a tangent plane to the tangent cone from the point and the planes through opposite edges of this circumscribed hexagon have a common line of intersection, the ray to the Brianchon point.

Ex. 1. Analytically we may consider the quadric $yz = wx$, and take the odd sides of the form (1) $x = \lambda_1 y$, $z = \lambda_1 w$, and the even (2) $x = \lambda_2 z$, $y = \lambda_2 w$. These two lines meet in the point whose coordinates are proportional to $\lambda_1\lambda_2$, λ_2, λ_1, 1, and the equation of the tangent plane at it is

$$t_{12} = x - \lambda_1 y - \lambda_2 z + \lambda_1\lambda_2 w = 0.$$

The Brianchon point will then evidently be the intersection of the planes

$$x - \lambda_1 y - \lambda_4 z + \lambda_1\lambda_4 w = 0,$$
$$x - \lambda_5 y - \lambda_2 z + \lambda_5\lambda_2 w = 0,$$
$$x - \lambda_3 y - \lambda_6 z + \lambda_3\lambda_6 w = 0,$$

its equation therefore is

$$\begin{vmatrix} a, & b, & c, & d \\ 1, & -\lambda_1, & -\lambda_4, & \lambda_1\lambda_4 \\ 1, & -\lambda_5, & -\lambda_2, & \lambda_5\lambda_2 \\ 1, & -\lambda_3, & -\lambda_6, & \lambda_3\lambda_6 \end{vmatrix} = 0,$$

and the equation of what we call the Pascal plane, may be written

$$\begin{vmatrix} x, & y, & z, & w \\ \lambda_1\lambda_4, & \lambda_4, & \lambda_1, & 1 \\ \lambda_5\lambda_2, & \lambda_2, & \lambda_5, & 1 \\ \lambda_3\lambda_6, & \lambda_6, & \lambda_3, & 1 \end{vmatrix} = 0,$$

if we multiply this by

$$\begin{vmatrix} 1, & -\lambda_1, & -\lambda_2, & \lambda_1\lambda_2 \\ 1, & -\lambda_5, & -\lambda_4, & \lambda_5\lambda_4 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix},$$

i.e. by $\lambda_1 - \lambda_5$, the result is

$$\begin{vmatrix} t_{12}, & 0, & 0, & (\lambda_3 - \lambda_1)(\lambda_6 - \lambda_2) \\ t_{45}, & 0, & 0, & (\lambda_3 - \lambda_5)(\lambda_6 - \lambda_4) \\ z, & \lambda_1, & \lambda_5, & \lambda_3 \\ w, & 1, & 1, & 1 \end{vmatrix},$$

hence, the value of the determinant is (compare *Conics*, p. 383)

$$(\lambda_3 - \lambda_5)(\lambda_6 - \lambda_4)\, t_{12} - (\lambda_3 - \lambda_1)(\lambda_6 - \lambda_2)\, t_{45},$$

with similar forms in t_{23}, t_{56} and in t_{34}, t_{61}: showing that the plane contains the lines t_{12}, t_{45}, and these other two lines.

Also since for *any* undetermined quantities x, y, z, w

$$\begin{Vmatrix} x, & y, & z, & w \\ \lambda_1\lambda_2, & \lambda_2, & \lambda_1, & 1 \\ \lambda_4\lambda_5, & \lambda_4, & \lambda_5, & 1 \end{Vmatrix} \begin{Vmatrix} 1, & -\lambda_1, & -\lambda_4, & \lambda_1\lambda_4 \\ 1, & -\lambda_5, & -\lambda_2, & \lambda_5\lambda_2 \\ 1, & -\lambda_3, & -\lambda_6, & \lambda_3\lambda_6 \end{Vmatrix}$$

$$= \begin{vmatrix} t_{14}, & t_{52}, & t_{36} \\ 0, & 0, & (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_6) \\ 0, & 0, & (\lambda_4 - \lambda_6)(\lambda_5 - \lambda_3) \end{vmatrix} = 0,$$

every point $xyzw$ is coplanar with the three points 1, 2; 4, 5; and that whose coordinates are the determinants in the second matrix. Therefore these last three points must be collinear; which is a verification that the diagonals in our hexagon intersect.

Ex. 2. The four faces of a tetrahedron pass each through a fixed point. Find the locus of the vertex if the three edges which do not pass through it move each in a fixed plane.

The locus is in general a surface of the third degree having the intersection of the three planes for a double point. It reduces to a cone of the second degree when the four fixed points lie in one plane.

Ex. 3. Find the locus of the vertex of a tetrahedron, if the three edges which pass through that vertex each pass through a fixed point, if the opposite face also pass through a fixed point and the three other vertices move in fixed planes. [A surface of the third degree.]

Ex. 4. A plane passes through a fixed point, and the points where it meets three fixed lines are joined by planes, each to one of three other fixed lines; find the locus of the intersection of the joining planes. [A surface of the third degree.]

Ex. 5. The sides of a polygon in space pass through fixed points, and all the vertices but one move in fixed planes; find the curve locus of the remaining vertex. [A twisted cubic.]

Ex. 6. State and solve the reciprocals of the problems of Exs. 2-5.

[Ex. 7. By projecting a skew hexagon drawn on a quadric from a point, reduce the Brianchon theorem for quadrics from the corresponding plane theorem.]

Projective Coordinates, Collineation, Reciprocation.

[144*c*. If x_1, x_2, x_3, x_4 are linear functions, with constant coefficients, of the Cartesian coordinates of a point, the position of the point is determined by the ratios of those four quantities, which are *homogeneous coordinates* of the point. Fiedler * shows that the three ratios involved may be defined by means of anharmonic ratios. Let A_1, A_2, A_3, A_4, E be five fixed points, no four of which lie in a plane. Let $A_1A_2 . A_3A_4EP$ signify the anharmonic ratio of the planes joining the line A_1A_2 to the points A_3, A_4, E, P. Then the position of the point P is uniquely determined if we know the three anharmonic ratios $A_1A_2 . A_3A_4EP$, $A_2A_3 . A_4A_1EP$, $A_3A_1 . A_2A_4EP$. Let e_1, e_2, e_3, e_4 be the distances of the point E, and p_1, p_2, p_3, p_4 the distances of the point P, from the planes $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ (more generally e_i and p_i are distances measured in the same direction to the planes involved). Then the anharmonic ratios are $\frac{p_3 : e_3}{p_4 : e_4}$, $\frac{p_4 : e_4}{p_1 : e_1}$, $\frac{p_2 : e_2}{p_4 : e_4}$, and are therefore defined by the

* Fourth German edition (1898) of the present work, Art. 52, Vol. I.

ratios $x_1 : x_2 : x_3 : x_4$ where $x_r \equiv \frac{p_r}{e_r}$. x_1, x_2, x_3, x_4 are called *projective coordinates* of P, for a reason which will shortly appear. The coordinates of E are 1, 1, 1, 1; it is called the *unit point*, and its position with respect to the tetrahedron of reference $A_1 A_2 A_3 A_4$ determines the scale of measurement. Any system of homogeneous coordinates may be expressed as projective coordinates by choosing E suitably; for example, if E is the centre of the inscribed sphere we get the ordinary perpendicular coordinates.

By translating these definitions into reciprocal language, we define in like manner the *projective coordinates of a plane.*

Projection (or linear projection) may be defined in general as a one-to-one correspondence (expressed by linear equations) between two classes of elements of space, or, as it is sometimes described—between elements of two spaces. If the elements in both classes are points, or in both classes are planes, the projection is called a *collineation;* if the elements of one class are points and of the other planes, it is called a *reciprocity.* Let us consider a collineation of points in one space. If $\xi_1, \xi_2, \xi_3, \xi_4$ are the coordinates of the projection of x_1, x_2, x_3, x_4, then by definition we have linear relations of the form $\rho\xi_1 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$, $\rho\xi_2 = \mu_1 x_1 +$ &c., $\rho\xi_3 = \nu_1 x_1 +$ &c., $\rho\xi_4 = \rho_1 x_1 +$ &c.

Now projective coordinates have the following property which justifies their name. The projective coordinates of a point (P) referred to the five points A_1, A_2, A_3, A_4, E, are the same as the projective coordinates of the projection (Q) of P referred to the projections B_1, B_2, B_3, B_4, F, of the five points, F being of course the new unit-point. To express it otherwise, *the projective coordinates of a point are equal to the projective coordinates of the projected point in the projected space.* To prove this we use the principle that the anharmonic ratio of a pencil of planes is equal to that of the projected pencil. For the anharmonic ratio of $aL + M, bL + M, cL + M, dL + M = 0$ is unaltered by linear transformation; we notice also that a point on a plane L becomes a point on the projected plane.

Projection might also be defined as a general transformation of points or planes—expressible algebraically—by which straight lines are transformed into straight lines. It follows at once from this definition that ruled surfaces project into ruled surfaces, and, in particular, quadrics into quadrics. Thus a quadric is defined projectively as the locus of points on a straight line meeting three fixed straight lines. A straight line has two "defining elements" of the same class (two points or two planes), thus a cone and a system of straight lines generating a plane curve, may be projectively defined by the same property, viz. all their generators have one defining element in common.

A projection is determined if we are given five non-special elements in one class and their correspondents in the other. This is clear from the fact that we require fifteen linear equations to determine the ratios of the sixteen constants λ_1, μ_1, &c. The geometrical meaning is that the projection is determined if we are given the elements A_1, A_2, A_3, A_4, E and their correspondents B_1, B_2, B_3, B_4, F, for then the coordinates of corresponding points are the same, each being referred to its own system. The algebraic treatment of collinea-

tion between planes, and of reciprocity between points and planes is exactly the same as that of the point-collineations which we have considered.]*

The Projective Theory of Distance and Angle.

[144*d*. Metrical geometry is concerned with actual distances and with the magnitude of angles, whereas projective geometry is concerned mainly with anharmonic ratios. It is clear that distances and the magnitude of angles are not preserved in the real projection of ordinary (or "Euclidean") space into itself, whereas projective properties remain unchanged; thus the method of projective geometry is far more general than that of metrical. Now the question arises, is it possible to define distance projectively, i.e. by means of anharmonic ratios, in such a way that the resulting conception will be consistent with, though more general than, the Euclidean conception, which will then present itself as a special case?

Now there are certain types of linear transformation which leave Euclidean (or ordinary) distances unchanged, namely the transformations corresponding to the motion of a rigid body combined with reflection with regard to any number of planes. The original axes being rectangular, the projection can evidently be expressed by the equations (Arts. 16, 17) by which we pass from one set of rectangular axes to another having a different origin; in this case however the new coordinates are supposed to represent the coordinates of a new point referred to the original axes. Let Ω represent the expression $u(x^2 + y^2 + z^2) + w^2$. Let the tetrahedron of reference be formed by three rectangular planes (x, y, z) and the plane at infinity (w); then w is constant in finite space. If u be supposed to evanesce towards zero, the transformation spoken of is exactly equivalent to a projection which is defined by the condition that the projection of a point for which $\Omega = 0$ also satisfies $\Omega = 0$; the constants of the projection being supposed finite.

To prove this let ξ, η, ζ, ω be the coordinates of Q, the projection of P (x, y, z, w). Then

$$x = \lambda_1\xi + \lambda_2\eta + \lambda_3\zeta + \lambda_4\omega$$

$$y = \mu_1\xi + \text{\&c.},\ z = \nu_1\xi + \text{\&c.},\ w = \rho_1\xi + \text{\&c.},$$

and the condition after removing an arbitrary multiplier is $\Omega_P \equiv \Omega_Q$, or

$$u(x^2 + y^2 + z^2) + w^2 \equiv u(\xi^2 + \eta^2 + \zeta^2) + \omega^2.$$

Equating coefficients, it follows that $\rho_4^2 = 1 - u(\lambda_4^2 + \mu_4^2 + \nu_4^2) = 1 - au$, say, where a is finite. Also $\rho_1 = -\dfrac{u(\lambda_1\lambda_4 + \mu_1\mu_4 + \nu_1\nu_4)}{\rho_4} = \dfrac{a_1 u}{(1 - au)^{\frac{1}{2}}}$, where a_1 is finite. Similarly $\rho_2 = \dfrac{a_2 u}{(1 - au)^{\frac{1}{2}}}$, $\rho_3 = \dfrac{a_3 u}{(1 - au)^{\frac{1}{2}}}$, where a_2, a_3 are finite. Hence

$$u(\lambda_1^2 + \mu_1^2 + \nu_1^2) + \frac{a_1^2 u^2}{1 - au} = u$$

and $u(\lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3) + \dfrac{a_2 a_3 u^2}{1 - au} = 0.$

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* For a fuller discussion see Fiedler, *op. cit.* Vol. I. Chaps. iv. and viii.

These equations hold whatever u may be and therefore in the limiting case corresponding to $u = 0$ we must have *

$$\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1, \text{ and } \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 = 0$$
$$\text{similarly } \lambda_2^2 + \mu_2^2 + \nu_2^2 = 1, \text{ and } \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 = 0$$
$$\lambda_3^2 + \mu_3^2 + \nu_3^2 = 1, \text{ and } \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0$$

whereas the quantities λ_4, μ_4, ν_4 are quite arbitrary, and $\rho_4^2 = 1$. If we replace $\frac{x}{w}, \frac{y}{w}, \frac{z}{w}$ by x, y, z and $\frac{\xi}{\omega}, \frac{\mu}{\omega}, \frac{\zeta}{\omega}$ by ξ, η, ζ, it is obvious that the projection represents the most general type of motion of a rigid body combined with reflection, and it is therefore described as a *congruent Euclidean transformation.* The quadric Ω may be taken to represent the degenerate quadric formed by tangent lines to the imaginary circle at infinity. For its tangential equation is $\frac{\alpha^2 + \beta^2 + \gamma^2}{u} + \delta^2 = 0$ and this is equivalent to $\alpha^2 + \beta^2 + \gamma^2 = 0$, δ being an arbitrary finite quantity. But the imaginary circle is the intersection of $x^2 + y^2 + z^2 = 0$ with $w = 0$ and its tangential equation is therefore $\alpha^2 + \beta^2 + \gamma^2 = 0$. The quadric $\Omega = 0$ is termed by Cayley the *Absolute*, but it is more convenient to call it the Euclidean Absolute. The result we have reached is that *distances and the magnitudes of angles are preserved in all projections which carry the Euclidean Absolute into itself.*

The interest of this somewhat artificial proposition springs from the generalisations of which it is capable, when the imaginary quadric Ω is replaced by any quadric real or imaginary; i.e. we may choose the Absolute arbitrarily, but the generalisation does not work elegantly unless the Absolute is either an imaginary quadric or a real un-ruled quadric, the condition required being that all straight lines in a certain region should meet the Absolute in imaginary points, or all in real points.

Let A, B be fixed points on the line containing the points P, Q, R. Then, if $[APBQ]$ signifies the anharmonic ratio of the range, we have $\log [APBQ] + \log [AQBR] = \log [APBR]$, which, we observe, is a relation of the same *form* as $PQ + QR = PR$, which is satisfied by Euclidean distances. Assuming that there are two fixed points on every line, and that the fixed points on one line project into the fixed points on the projected line, the preceding relation connecting the anharmonic ratio will evidently be unchanged by projection, since these ratios are unchanged. It is natural to take for the two fixed points on a line the points where it meets a fixed quadric $U = 0$. For a quadric can be defined projectively (Art. 144c), and therefore the conditions required are satisfied. The generalised or non-Euclidean distance between two points is then defined as a constant multiple of *the logarithm of the anharmonic ratio of the range determined by the two points and the points where the joining line meets the Absolute* and distance is invariant for all projections which carry

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* These inferences are valid in spite of the fact that we have divided across by u; it is evident that, e.g. $\lambda_1^2 + \mu_1^2 + \nu_1^2$ approaches the limit 1 as u diminishes without limit. But we cannot expand and equate coefficients of u, since the coefficients of the projection vary with u, being conditioned only by the fact that they are *finite.*

the Absolute into itself. All projections with reference to a given Absolute are said to belong to the same group of *congruent* transformations. We shall see that Euclidean distance is a special case of projective distance when the Absolute is a sphere of infinite radius with its centre in finite space, hence the generalisation is valid and natural.

Assuming that the Absolute is either a real non-ruled quadric or an imaginary quadric with real coefficients, it can always be reduced to the form $\lambda\,(X^2 + Y^2 + Z^2) + W^2 = 0$, and putting $\frac{X}{W} = x$, $\frac{Y}{W} = y$, $\frac{Z}{W} = z$, we need only consider the form $\lambda\,(x^2 + y^2 + z^2) + 1 = 0$. There are three cases according as λ is positive, negative, or having zero as a limit. The corresponding types of metrical geometry are called *elliptic*, *hyperbolic* and *parabolic*, the Euclidean type being a form of the last.

When λ is positive, substituting $lx + mx'$, &c., in the equation $U = 0$ (x, y, z, 1, being the coordinates of P and x', y', z', 1, of Q), it is found without difficulty that the anharmonic ratio $[APBQ]$, which is equal to the ratio of the two values of $\frac{l}{m}$, is $e^{2i\theta}$ where

$$\cos\theta = \frac{\lambda(xx' + yy' + zz') + 1}{\{\lambda(x^2 + y^2 + z^2) + 1\}^{\frac{1}{2}}\,\{\lambda(x'^2 + y'^2 + z'^2) + 1\}^{\frac{1}{2}}}.$$

Let the distance $[PQ]$, assumed real, be equal to $\frac{k}{2i}\log\,[APBQ]$. Then $\cos\frac{[PQ]}{k}$ has the value of $\cos\theta$ just written and is thus expressed in terms of the coordinates of P and Q. For given values of these coordinates, there are two values of $[PQ]$ corresponding to the two signs of the radical, and the sum of these two values is πk. Hence the whole straight line is of finite length.

In general the space defined is elliptic if the Absolute is an imaginary quadric whose equations have real coefficients, and the value of $\cos\theta$ is $\frac{T}{\sqrt{UU'}}$, where $U = 0$ is the Absolute and $2T = \Sigma x'\frac{dU}{dx}$ for homogeneous coordinates. For all real points θ is real, and therefore the elliptic system of metrical geometry is applicable to the whole of space, space being defined as the assemblage of entities determined by coordinates whose values range from $+\infty$ to $-\infty$.

For *hyperbolic* space λ is negative and the Absolute is a real un-ruled quadric. Using a similar method, we find $[APBQ] = e^{2\theta}$, where $\cosh\theta = \frac{T}{\sqrt{UU'}}$, and if $[PQ] = \frac{1}{2}k\log\,[APBQ]$, we have $\cosh\frac{[PQ]}{k} = \frac{T}{\sqrt{UU'}}$. There is only one distance between the two points; if this is real U and U' must have the same sign and $\frac{T}{\sqrt{UU'}}$ must be > 1, i.e. the line PQ meets the Absolute in real points. These two conditions are expressed by saying that the points for which the formula is valid lie wholly within the Absolute. If Q coincides with A, $U' = 0$, and the distance $[PA]$ is therefore infinite, i.e. all

points for which this formula gives any finite value of $[PQ]$ are at an infinite distance (in the sense considered) from the Absolute.

If λ is indefinitely diminished we get *parabolic space*, which is a limiting case of the two preceding. If in the elliptic case we put $k^2\lambda = 1$ (or in the hyperbolic $k^2\lambda = -1$), we find by expansion (neglecting $\frac{1}{k^4}$, and using

$$\cos\theta = 1 - \frac{\theta^2}{2} \text{ or } \cosh\theta = 1 + \frac{\theta^2}{2}\Big),$$

$$[PQ]^2 = (x - x')^2 + (y - y') + (z - z')^2,$$

i.e. *Euclidean distance is defined projectively by taking for the Absolute the quadric generated by the tangent lines to the imaginary circle at infinity.*

Similarly the generalised *angle between two planes* is defined as a constant multiple of the logarithm of the anharmonic ratios of the pencil they form with the tangent planes to the Absolute through their line of intersection. Taking for example the elliptic type, and putting the constant multiple equal to $\frac{1}{2i}$ we find for the two planes $\alpha x + \beta y + \gamma z + \delta = 0$, $\Sigma\alpha' x + \delta' = 0$, that

$$\cos\phi = \frac{\Sigma\alpha\alpha' + \lambda\delta\delta'}{(\Sigma\alpha^2 + \lambda\delta^2)^{\frac{1}{2}}(\Sigma\alpha'^2 + \lambda\delta'^2)^{\frac{1}{2}}}$$

which gives the Euclidean formula when $\lambda = 0$.

In general if $\Sigma = 0$ is the tangential equation (Arts. 79, 80) of the Absolute, for elliptic space

$$\cos\phi = \frac{\Pi}{\sqrt{\Sigma\Sigma'}}.$$

The projective angle between two lines is the anharmonic ratio of the pencil they form with the tangent lines to the section of the Absolute by their plane. In the Euclidean case these tangent lines are lines to the circular points at infinity in the plane].*

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* For further particulars as to the generalised meaning of congruence, see Whitehead's *Axioms of Descriptive Geometry* (1907), where the method of Lie is followed.

CHAPTER VIII.

FOCI AND CONFOCAL QUADRICS.*

Foci and Focal Conics of a Quadric.

145. WHEN U represents a sphere, the equation of a quadric having double contact with it, $U = LM$ expresses, as at *Conics*, Art. 260, that the square of the tangent from any point on the quadric to the sphere is in a constant ratio to the rectangle under the distances of the same point from two fixed planes. The planes L and M are evidently parallel to the planes of circular section of the quadric, since they are planes of its intersection with a sphere; and their intersection is therefore parallel to an axis of the quadric (Arts. 103, 139). We have seen (*Conics*, Art. 261) that the focus of a conic may be considered as an infinitely small circle having double contact with the conic, the chord of contact being the directrix. In like manner *we may define a focus of a quadric as an infinitely small sphere having double contact with the quadric, the chord of contact being then the corresponding directrix*. That is to say, the point $\alpha\beta\gamma$ is a focus if the equation of the quadric can be expressed in the form

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \phi,$$

where ϕ is the product of the equations of two planes. We must discuss separately, however, the two cases, where these planes are real and where they are imaginary. In the one case the equation is of the form $U = LM$, in the other $U = L^2 + M^2$. In the first case, the directrix (the line LM) is parallel

* The properties treated of in this chapter were first studied in detail by Chasles and by MacCullagh, who about the same time independently arrived at the principal of them. Chasles' results will be found in the notes to his *Aperçu Historique*, published in 1837.

to that axis of the surface through which real planes of circular section can be drawn; for example, to the mean axis if the surface be an ellipsoid. In the second case the line LM is parallel to one of the other axes.

We can show directly that the line LM is parallel to an axis of the surface. For if the coordinate planes x and y be any two planes mutually at right angles passing through LM; then since L and M are both of the form $\lambda x + \mu y$, the quantities LM and $L^2 + M^2$ will be both of the form $ax^2 + 2hxy + by^2$. And, as in plane geometry, it is proved that by turning round the coordinate planes x and y, this quantity can be made to take the form $px^2 \pm qy^2$. The equations then, $U = LM$, $U = L^2 + M^2$, written in full, are of the form $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = px^2 \pm qy^2$, and since the terms yz, zx, xy do not enter into the equation, the axes of coordinates are parallel to the axes of the surface.

146. A focus of a plane curve has been defined (*Higher Plane Curves*, Art. 138) as the point of intersection of two tangents, passing each through one of the circular points at infinity. The definition just given of a focus of a quadric may be stated in an analogous form. When the origin is a focus we have just seen that the equation of the quadric may be written in the form $U = LM$, where U, or $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$, denotes a cone whose vertex is the focus, and which passes through the imaginary circle at infinity. The form of the equation shows (Art. 137) that this cone has double contact with the quadric in the points where the line LM meets it. The tangent plane to the surface at either point of contact will then be a tangent plane to the cone, and will therefore pass through a tangent line of the circle at infinity. *We may thus define a focus as a point through which can be drawn two lines σ, each touching the surface and meeting the imaginary circle at infinity, and such that the tangent plane to the surface through either also touches the circle at infinity.* This definition is not restricted to the case of a quadric, but applies to a surface of any order.

Starting from this definition, if we desire to find the foci of any surface, we should consider the tangent planes to the surface drawn through the tangent lines of the circle at infinity: these form a singly infinite series of planes, and will envelop a developable surface. The intersection of two consecutive such planes will be a line σ, and will be a generator of the developable. A focus, being a point through which pass two lines σ, that is to say, two generators of the developable, must be a double point on the developable. Now we shall see hereafter that a developable has in general a series of double points forming a nodal curve or curves; we infer, therefore, that the foci of a surface in general are not detached points, but a series of points forming a curve or curves. We shall show directly, in the next article, that this is so in the case of a quadric. It is evident from this definition that *two surfaces will have the same series of foci, if the developable, just spoken of, passing through the tangent lines of the circle at infinity and enveloping the surface, be common to both.* [The tangential equation of any one of a system of confocal quadrics is therefore $\Sigma + \lambda\Omega = 0$ where $\Sigma = 0$ is the tangential equation of one of them and $\Omega = 0$ the tangential equation of the circle at infinity.]

147. Let us then directly examine whether a given central quadric necessarily has a focus, and whether it has more than one. For greater generality, instead of taking the directrix for the axis of z, we take any parallel line, and the equation of the last article becomes

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = p(x-\alpha')^2 + q(y-\beta')^2;\,^*$$

and we are about to inquire whether any values can be assigned to α, β, γ, α', β', p, q, which will make this identical with a given equation

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1.$$

* When p and q have opposite signs the planes of contact of the focus with the quadric are real, while they are imaginary when p and q have the same sign.

Now first, in order that the origin may be the centre, we have $\gamma=0$, $\alpha=p\alpha'$, $\beta=q\beta'$: by the help of which equations, eliminating α', β', the form written above becomes

$$(1-p)x^2+(1-q)y^2+z^2=\frac{1-p}{p}\alpha^2+\frac{1-q}{q}\beta^2,$$

whence $\quad 1-p=\frac{C}{A},\ p=\frac{A-C}{A};\ 1-q=\frac{C}{B},\ q=\frac{B-C}{B};$

$$\frac{1-p}{p}\alpha^2+\frac{1-q}{q}\beta^2=C,$$

or

$$\frac{\alpha^2}{A-C}+\frac{\beta^2}{B-C}=1.$$

Thus it appears that the surface being given, the constants p and q are determined, but that the focus may lie anywhere on the *conic*

$$\frac{\alpha^2}{A-C}+\frac{\beta^2}{B-C}=1,$$

which accordingly is called a *focal conic* of the surface.

Since we have purposely said nothing as to either the signs or the relative magnitudes of the quantities A, B, C, it follows that *there is a focal conic in each of the three principal planes, and also that this conic is confocal with the corresponding principal section of the surface;* the conics

$$\frac{\alpha^2}{A}+\frac{\beta^2}{B}=1,\ \frac{\alpha^2}{A-C}+\frac{\beta^2}{B-C}=1,$$

being plainly confocal. Any point $\alpha\beta$ on a focal conic being taken for focus, the corresponding directrix is a perpendicular to the plane of the conic drawn through the point

$$\alpha'=\frac{\alpha}{p},\ \beta'=\frac{\beta}{q},\ \text{or}\ \alpha'=\frac{A\alpha}{A-C},\ \beta'=\frac{B\beta}{B-C}.$$

These values may be interpreted geometrically by saying that *the foot of the directrix is the pole, with respect to the principal section of the surface, of the tangent to the focal conic at the point $\alpha\beta$.* For this tangent is

$$\frac{\alpha x}{A-C}+\frac{\beta y}{B-C}=1,\ \text{or}\ \frac{\alpha' x}{A}+\frac{\beta' y}{B}=1,$$

which is manifestly the polar of $\alpha'\beta'$ with regard to $\frac{x^2}{A}+\frac{y^2}{B}=1$.

Ex. 1. The line joining any focus to the foot of the corresponding directrix is normal to the focal conic.

Ex. 2. The feet of the directrices must evidently lie on that conic which is the locus of the poles of the tangents of the focal conic with regard to the corresponding principal section of the quadric. The equation of this conic is

$$x^2\frac{A-C}{A^2}+y^2\frac{B-C}{B^2}=1;$$

for if we eliminate α, β from the equation of the focal conic and the equations connecting $\alpha\beta$, $\alpha'\beta'$, we obtain this relation to be satisfied by the latter pair of coordinates. The directrices themselves form a cylinder of which the conic just written is the base.

148. Let us now examine in detail the different classes of central surfaces, in order to investigate the nature of their focal conics and to find to which of the two different kinds of foci the points on each belong. It is plain that the equation

$$\frac{\alpha^2}{A-C}+\frac{\beta^2}{B-C}=1$$

will represent an ellipse when C is algebraically the least of the three quantities A, B, C; a hyperbola when C is the middle, and will become imaginary when C is the greatest.

Of the three focal conics therefore of a central quadric, one is always an ellipse, one a hyperbola, and one imaginary. In the case of the ellipsoid, for example, the equations of the focal ellipse and focal hyperbola are respectively

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1.$$

The corresponding equations for the hyperboloid of one sheet are found by changing the sign of c^2, and those for the hyperboloid of two sheets by changing the sign both of b^2 and c^2.

Further, we have seen that foci belong to the class whose planes of contact are imaginary, or are real, according as p and q have the same or opposite signs, and that

$$p=(A-C):A,\ q=(B-C):B.$$

Now if C be the least of the three, in these fractions both numerators are positive, and the denominators are also positive in the case of the ellipsoid and hyperboloid of one sheet, but in the case of the hyperboloid of two sheets one of the denominators is negative. Hence *the points on the focal*

ellipse are foci of the class whose planes of contact are imaginary in the cases of the ellipsoid and of the hyperboloid of one sheet, but of the opposite class in the case of the hyperboloid of two sheets. Next, let C be the middle of the three quantities; then the two numerators have opposite signs, and the denominators have the same sign in the case of the ellipsoid, but opposite signs in the case of either hyperboloid. Hence *the points of the focal hyperbola belong to the class whose planes of contact are real in the case of the ellipsoid, and to the opposite class in the case of either hyperboloid.* It will be observed then that *all* the real foc of the hyperboloid of one sheet belong to the class whose planes of contact are imaginary; but that the focal conics of the other two surfaces contain foci of opposite kinds, the ellipse of the ellipsoid and the hyperbola of the hyperboloid being those whose planes of contact are imaginary. This is equivalent to what appeared (Art. 145) that foci having real planes of contact can only lie in planes perpendicular to that axis of a quadric through which real planes of circular section can be drawn.

149. *Focal conics with real planes of contact intersect the surface in real points, while those of the other kind do not.* In fact, if the equation of a surface can be thrown into the form $U = L^2 + M^2$, and if the coordinates of any point on the surface make $U = 0$, they must also make $L = 0$, $M = 0$; that is to say, the focus must lie on the directrix. But in this case the surface could only be a cone. For taking the origin at the focus, the equation $x^2 + y^2 + z^2 = L^2 + M^2$, where L and M each pass through the origin, would contain no terms except those of the highest degree in the variables, and would therefore represent a cone (Art. 66).

The focal conic on the other hand, which consists of foci of the first kind, passes through the [real] umbilics. For if the equation of the surface can be thrown into the form $U = LM$, and the coordinates of a point on the surface make $U = 0$, they must also make either L or $M = 0$. But since the sur-

face passes through the intersection of U, L; if the point U lies on L, the plane L intersects the surface in an infinitely small circle; that is to say, is a tangent at an umbilic.

From the fact that focal conics which consist of foci having real planes of contact pass through the [real] umbilics, MacCullagh gave them the name *umbilicar* focal conics.

150. *The section of the quadric by a plane passing through a focus and the corresponding directrix is a conic having the same point and line for focus and directrix.* For, taking the origin at the focus, the equation is either $x^2 + y^2 + z^2 = LM$, or $x^2 + y^2 + z^2 = L^2 + M^2$. And if we make $z = 0$, the equation of the section is $x^2 + y^2 = lm$ or $= l^2 + m^2$, where l, m are the sections of L, M by the plane z. But if this plane pass through LM, these sections coincide, and the equation reduces to $x^2 + y^2 = l^2$, which represents a conic having the origin for the focus and l for the directrix. Since the plane joining the focus and directrix is normal to the focal conic (Art. 147), we may state the theorem just proved, as follows: *Every plane section normal to a focal conic has for a focus the point where its plane is normal to the focal conic.*

151. If the given quadric were a cone $\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0$, the reduction of the equation to the form $U = L^2 \pm M^2$ proceeds exactly as before, and it is proved that the coordinates of the focus must fulfil the condition $\frac{\alpha^2}{A - C} + \frac{\beta^2}{B - C} = 0$, which represents either two right lines or an infinitely small ellipse, according as $A - C$ and $B - C$ have opposite or the same signs. In other words, *in the case of the cone the focal hyperbola becomes two right lines, while the focal ellipse contracts to the vertex of the cone.* For the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, the equation of the focal lines is $\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 + c^2} = 0$.

The focal lines of the cone, asymptotic to any hyperboloid, are plainly the asymptotes to the focal hyperbola of the surface.

The foci on the focal lines are all of the class whose planes of contact are imaginary; but the vertex itself, besides being in two ways a focus of this kind, may also be a focus of the other kind, for the equation of the cone just written takes any of the three forms

$$x^2+y^2+z^2=\frac{a^2+c^2}{a^2}x^2+\frac{b^2+c^2}{b^2}y^2,$$

or
$$=\frac{a^2-b^2}{a^2}x^2+\frac{b^2+c^2}{c^2}z^2, \text{ or} = -\frac{b^2-a^2}{b^2}y^2+\frac{a^2+c^2}{c^2}z^2.$$

The directrix, which corresponds to the vertex considered as a focus, passes through it.

The line joining any point on a focal line to the foot of the corresponding directrix is perpendicular to that focal line. This follows as a particular case of what has been already proved for the focal conics in general, but may also be proved directly. The coordinates of the foot of the directrix have been proved to be $\alpha'=\frac{A\alpha}{A-C}$, $\beta'=\frac{B\beta}{B-C}$, the equation of the line joining this point to $\alpha\beta$ is

$$\frac{\beta}{B-C}x-\frac{\alpha}{A-C}y=\alpha\beta\left(\frac{1}{B-C}-\frac{1}{A-C}\right),$$

and the condition that this should be perpendicular to the focal line $\beta x=\alpha y$ is $\frac{\alpha^2}{A-C}+\frac{\beta^2}{B-C}=0$, which we have already seen is satisfied.

In like manner, as a particular case of Art. 150, *the section of a cone by a plane perpendicular to either of its focal lines is a conic of which the point in the focal line is a focus.* The focal lines of this article are therefore identical with those defined (Art. 125).

152. *The focal lines of a cone are perpendicular to the circular sections of the reciprocal cone* (see Art. 125).

For the circular sections of the cone $Ax^2+By^2+Cz^2=0$, are (see Art. 103) parallel to the planes

$$(A-C)\,x^2+(B-C)\,y^2=0,$$

and the corresponding focal lines of the reciprocal cone

$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0$, are, as we have just seen, $\frac{x^2}{A-C}+\frac{y^2}{B-C}=0$, and the lines represented by the latter equation are evidently perpendicular to the planes represented by the former.

153. The investigation of the foci of the other species of quadrics proceeds in like manner. Thus for the *paraboloids* included in the equation $\frac{x^2}{A}+\frac{y^2}{B}=2z$; this equation can be written in either of the forms

$$(x-a)^2+y^2+(z-\gamma)^2=\frac{A-B}{A}\left(x-\frac{A}{A-B}a\right)^2+(z-\gamma+B)^2,$$

where
$$\frac{a^2}{A-B}=2\gamma-B,$$

or
$$x^2+(y-\beta)^2+(z-\gamma)^2=\frac{B-A}{B}\left(y-\frac{B}{B-A}\beta\right)^2+(z-\gamma+A)^2,$$

where
$$\frac{\beta^2}{B-A}=2\gamma-A.$$

It thus appears that *a paraboloid has two focal parabolas, which may easily be seen to be each confocal with the corresponding principal section.* The focus belongs to one or other of the two kinds already discussed, according to the sign of the fraction $(A-B):A$. In the case of the elliptic paraboloid, therefore, where both A and B are positive, if A be the greater, then the foci in the plane xz are of the class whose planes of contact are imaginary, while those in the plane yz are of the opposite class. But since if we change the sign either of A or of B, the quantity $(A-B):A$ remains positive, we see that *all* the foci of the hyperbolic paraboloid belong to the former class, a property we have already seen to be true of the hyperboloid of one sheet.

It remains true that the line joining any focus to the foot of the corresponding directrix is normal to the focal curve, and that the foot of the directrix is the pole with regard to the principal section of the tangent to the focal conic. The feet of the directrices lie on a parabola, and the directrices themselves generate a parabolic cylinder.

To complete the discussion it remains to notice the foci of the different kinds of cylinders, but it is found without the slightest difficulty that when the base of the cylinder is an ellipse or hyperbola there are two focal lines; namely, lines drawn through the foci of the base parallel to the generators of the cylinder; while, if the base of the cylinder is a parabola, there is one focal line passing in like manner through the focus of the base.

154. The geometrical interpretation of the equation $U = LM$ has been already given. We learn from it this property of foci whose planes of contact are real, that *the square of the distance of any point on a quadric from such a focus is in a constant ratio to the product of the perpendiculars let fall from the point on the quadric, on two planes drawn through the corresponding directrix, parallel to the real planes of circular section.* The corresponding property of foci of the other kind, which is less obvious, was discovered by MacCullagh. It is, that *the distance of any point on the quadric from such a focus is in a constant ratio to its distance from the corresponding directrix, the latter distance being measured parallel to either of the planes of real circular section.*

Suppose, in fact, we try to express the distance of the point $x'y'z'$ from a directrix parallel to the axis of z and passing through the point whose x and y are α', β', the distance being measured parallel to a directive plane $z = mx$. Then a parallel plane through $x'y'z'$, viz. $z - z' = m(x - x')$, meets the directrix in a point whose x and y of course are α', β', while its z is given by the equation $z - z' = m(\alpha' - x')$. The square of the distance required is therefore

$$(x' - \alpha')^2 + (y' - \beta')^2 + m^2(x' - \alpha')^2 = (y' - \beta')^2 + (1 + m^2)(x' - \alpha')^2.$$

In the equation then of Art. 147,

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = p(x - \alpha')^2 + q(y - \beta')^2,$$

where p and q are both positive, and p is supposed greater than q, the right-hand side denotes q times the square of the distance of the point on the quadric from the directrix, the distance being measured parallel to the plane $z = mx$ where

$m^2 = (p - q) : q$. By putting in the values of p and q, given in Art. 147, it may be seen that this is a plane of circular section, but it is evident geometrically that this must be the case. For consider the section of the quadric by any plane parallel to the directive plane; then, since evidently the distances of every point in such a section are measured from the same point on the directrix, the distance of every point in the section from this fixed point is in a constant ratio to its distance from the focus. But when the distances of a variable point from two fixed points have to each other a constant ratio, the locus is a sphere. The section therefore is the intersection of a plane and a sphere; that is, a circle.

An exception occurs when the distance from the focus is to be *equal* to the distance from the directrix. Since the locus of a point equidistant from two fixed points is a plane, it appears as before, that in this case the sections parallel to the directive plane are right lines. By referring to the previous articles, it will be seen (see Art. 153) that the ratio we are considering is one of equality ($q = 1$) only in the case of the hyperbolic paraboloid, a surface which the directive plane could not meet in circular sections, seeing that it has not got any. MacCullagh calls the ratio of the focal distance to that from the directrix, the modulus of the surface, and the foci having imaginary planes of contact, he calls modular foci.*

155. It was observed (Art. 137) that any two quadrics of the form $U - LM = 0$ and $U = 0$, are enveloped by two cones,

* In the year 1836 MacCullagh published this modular method of generation of quadrics. In 1842 I published the supplementary property possessed by the non-modular foci. Not long after, Amiot (*Liouville*, VIII., p. 161, and X., p. 109) independently noticed the same property, but owing to his not being acquainted with MacCullagh's method of generation, Amiot failed to obtain the complete theory of the foci. MacCullagh has published a detailed account of the focal properties of quadrics, which will be found in the *Proceedings of the Royal Irish Academy*, Vol. II., p. 446: reprinted at p. 260 of his *Collected Works*, Dublin, 1880. Townsend also has published a valuable paper (*Cambridge and Dublin Mathematical Journal*, Vol. III., pp. 1, 97, 148) in which the properties of foci, considered as the limits of spheres having double contact with a quadric, are very fully investigated.

and when U represents a sphere, these are cones of revolution as every cone enveloping a sphere must be. Further, when U reduces to a point-sphere, these cones coincide in a single one, having that point for its vertex; and we may therefore infer that *the cone enveloping a quadric and having any focus for its vertex is one of revolution.*

This theorem being of importance, we give a direct algebraical proof of it. First, it will be observed, that any equation of the form $x^2 + y^2 + z^2 = (ax + by + cz)^2$ represents a right cone. For if the axes be transformed, remaining rectangular, but so that the plane denoted by $ax + by + cz$ may become one of the coordinate planes, the equation of the cone will become $X^2 + Y^2 + Z^2 = \lambda X^2$, which denotes a cone of revolution, since the coefficients of Y^2 and Z^2 are equal.

But now if we form, by the rule of Art. 78, the equation of the cone whose vertex is the origin and circumscribing $x^2 + y^2 + z^2 - L^2 - M^2$, where

$$L = ax + by + cz + d,\ M = a'x + b'y + c'z + d',$$

it is found to be

$$(d^2 + d'^2)\,(x^2 + y^2 + z^2 - L^2 - M^2) + (dL + d'M)^2 = 0,$$

or

$$(d^2 + d'^2)\,(x^2 + y^2 + z^2) - (d'L - dM)^2 = 0,$$

which we have seen represents a right cone.

[The method of Art. 119 can also be used to prove that the locus of vertices of right tangent cones consists of the focal conics. But if we do not assume that f, g, or h vanish, the locus found, by method of Art. 118, reduces to the quadric itself.]

Cor. Since, in spherical reciprocation, the circumscribing cone whose vertex is the origin corresponds to the asymptotic cone of the reciprocal surface, it follows from this article, that *the reciprocal of a quadric with regard to any focus is a surface of revolution.*

A few additional properties of foci easily deduced from the principles laid down are left as an exercise to the reader.

Ex. 1. The polar of any directrix is the tangent to the focal conic at the corresponding focus.

Ex. 2. The polar plane of any point on a directrix is perpendicular to the line joining that point to the corresponding focus.

Ex. 3. If a line be drawn through a fixed point O cutting any directrix of a quadric, and meeting the quadric in the points A, B; then if F be the corresponding focus, $\tan \frac{1}{2}AFO \cdot \tan \frac{1}{2}BFO$ is constant. This is proved as the corresponding theorem for plane conics. *Conics*, Art. 226, Ex. 8.

Ex. 4. This remains true if the point O move on any other quadric having the same focus, directrix, and planes of circular section.

Ex. 5. If two such quadrics be cut by any line passing through the common directrix, the angles subtended at the focus by the intercepts are equal.

Ex. 6. If a line through a directrix touch one of the quadrics, the chord intercepted on the other subtends a constant angle at the focus.

156. The product of the perpendiculars from the two foci of a surface of revolution round the transverse axis, on any tangent plane, is evidently constant. Now if we reciprocate this property with regard to any point by the method used in Art. 126, we learn that the square of the distance from the origin of any point on the reciprocal surface is in a constant ratio to the product of the distances of the point from two fixed planes.

It appears from Art. 126, Ex. 5, that the two planes are planes of circular section of the asymptotic cone to the new surface, and therefore of the new surface itself. The intersection of the two planes is the reciprocal of the line joining the two foci; that is to say, of the axis of the surface of revolution. The property just proved* belongs as we know (Art. 154) to every point on the umbilicar focal conic; hence the reciprocal of any quadric with regard to an umbilicar focus, is a surface of revolution round the transverse axis: but with regard to a modular focus is a surface of revolution round the conjugate axis.

By reciprocating properties of quadrics of revolution, we obtain properties of any quadric with regard to focus and corresponding directrix. It is to be noted, that the axis of the figure of revolution of either kind is the reciprocal of the directrix corresponding to the given focus; and is parallel to the tangent to the focal conic at the given focus (see Art. 147).

* It was in this way I was first led to this property, and to observe the distinction between the two kinds of foci.

The left-hand column contains properties of surfaces of revolution, the right-hand of quadrics in general.

Ex. 1. The tangent cone whose vertex is any point on the axis is a right cone whose tangent planes make a constant angle with the plane of contact, which plane is perpendicular to the axis.

The cone whose vertex is a focus and base any section whose plane passes through the corresponding directrix, is a right cone, whose axis is the line joining the focus to the pole of the plane of section, and this right line is perpendicular to the plane through focus and directrix.

Ex. 2. Any tangent plane is at right angles to the plane through the point of contact and the axis.

The line joining a focus to any point on the surface is at right angles to the line joining the focus to the point where the corresponding tangent plane meets the directrix.

Ex. 3. The polar plane of any point is at right angles to the plane containing that point and the axis.

The line joining a focus to any point is at right angles to the line joining the focus to the point where the polar plane meets the directrix.

Ex. 4. Any two conjugate lines are such that the planes joining them to the focus are at right angles. (Ex. 7, Art. 126).

Any two conjugate lines pierce a plane through a directrix parallel to circular sections, in two points which subtend a right angle at the corresponding focus.

Ex. 5. If a cone circumscribe a surface of revolution, one principal plane is the plane of vertex and axis.

The cone whose base is *any* plane section of a quadric and vertex any focus has for one axis the line joining focus to the point where the plane meets the directrix.

Ex. 6. The cone whose vertex is a focus and base any plane section is a right cone. (Ex. 2, Art. 126).

The cone is a right cone whose vertex is a focus and base the section made by any tangent cone on a plane through the corresponding directrix parallel to those of the circular sections.

[*Note.*—Foci, as we have seen, are spheres of zero radius having double contact with the quadric. We may investigate the more general question of *spheres of given radius having double contact with a quadric.* If the sphere S be $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2=0$, and the quadric U be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$, then if S has double contact with U, it is possible to find λ such that $S+\lambda U=0$ represents a pair of planes. Equating to zero the discriminant of $S+\lambda U$ we find $\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}-1-\frac{r^2}{\lambda}=0$. Expressing that the terms of the

highest degree in $S+\lambda U$ break into factors, we find $\lambda = -a^2$, or $\lambda = -b^2$, or $\lambda = -c^2$. Hence the *locus of the centres of the spheres consists of three conics in the principal planes;* for example $\lambda = -c^2$ gives $\gamma = 0$ and

$$\frac{\alpha^2}{a^2-c^2}+\frac{\beta^2}{b^2-c^2}=1-\frac{r^2}{c^2}.$$

The focal conics are special cases of these. We can classify these loci and determine for different values of r and for the different species of quadrics whether they are ellipses, or hyperbolas, real or imaginary.]

Focal Conics and Confocal Quadrics.

157. In the preceding section an account has been given of the relations which each focus of a quadric considered separately bears to the surface. We shall in this section give an account of the properties of the conics which are the assemblage of foci, and of the properties of confocal surfaces. And we commence by pointing out a method by which we should be led to the consideration of the focal conics of a quadric independently of the method followed in the last section.

Two concentric and coaxal conics are said to be confocal when the difference of the squares of the axes is the same for both. Thus given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, any conic is confocal with it whose equation is of the form

$$\frac{x^2}{a^2\pm\lambda^2}+\frac{y^2}{b^2\pm\lambda^2}=1.$$

If we give the positive sign to λ^2, the confocal conic will be an ellipse; it will also be an ellipse when λ^2 is negative as long as it is less than b^2. When λ^2 is between b^2 and a^2 the confocal curve is a hyperbola, and when λ^2 is greater than a^2 the curve is imaginary. If $\lambda^2=b^2$, the equation reducing itself to $y^2=0$, the axis of x itself is the limit which separates confocal ellipses from hyperbolas. But the two foci belong to this limit in a special sense. In fact, through a given point $x'y'$ can in general be drawn two conics confocal to a given one, since we have a quadratic to determine λ^2, viz.

$$\frac{x'^2}{a^2-\lambda^2}+\frac{y'^2}{b^2-\lambda^2}=1,$$

or $$\lambda^4-\lambda^2(a^2+b^2-x'^2-y'^2)+a^2b^2-b^2x'^2-a^2y'^2=0.$$

When $y'=0$ this quadratic becomes

$$(\lambda^2 - b^2)\ (\lambda^2 - a^2 + x'^2) = 0,$$

and *one* of its roots is $\lambda^2 = b^2$: but if we have also $x'^2 = a^2 - b^2$, the second root is also $\lambda^2 = b^2$, and therefore the two foci are in a special sense points corresponding to the value $\lambda^2 = b^2$. If in the equation $\frac{x^2}{a^2-\lambda^2} + \frac{y^2}{b^2-\lambda^2} = 1$, we make $\lambda^2 = b^2$, $\frac{y^2}{b^2-\lambda^2} = 0$, we get the equation of the two foci $\frac{x^2}{a^2-b^2} = 1$.

158. Now in like manner *two quadrics are said to be confocal if the differences of the squares of the axes be the same for both.* Thus given the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, any surface is confocal whose equation is of the form

$$\frac{x^2}{a^2 \pm \lambda^2} + \frac{y^2}{b^2 \pm \lambda^2} + \frac{z^2}{c^2 \pm \lambda^2} = 1.$$

If we give λ^2 the positive sign, or if we take it negative and less than c^2, the surface is an ellipsoid. A sphere of infinite radius is the limit of all ellipsoids of the system, being what the equation represents when $\lambda^2 = \infty$. When λ^2 is negative and between c^2 and b^2 the surface is a hyperboloid of one sheet. When it is between b^2 and a^2 it is a hyperboloid of two sheets.

When $\lambda^2 = c^2$ the surface reduces itself to the plane $z=0$, but if we make in the equation $\lambda^2 = c^2$, $\frac{z^2}{\lambda^2 - c^2} = 0$, the points on the conic thus found, viz. $\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1$, belong in a special sense to the limit separating ellipsoids and hyperboloids. In fact, in general *through any point $x'y'z'$ can be drawn three surfaces confocal to a given one;* for regarding λ^2 as the unknown quantity, we have evidently a cubic for the determination of it; namely,

$$\frac{x'^2}{a^2-\lambda^2} + \frac{y'^2}{b^2-\lambda^2} + \frac{z'^2}{c^2-\lambda^2} = 1,$$

or $x'^2(b^2-\lambda^2)\ (c^2-\lambda^2) + y'^2(c^2-\lambda^2)\ (a^2-\lambda^2) + z'^2(a^2-\lambda^2)\ (b^2-\lambda^2)$

$$= (a^2-\lambda^2)\ (b^2-\lambda^2)\ (c^2-\lambda^2).$$

If $z'=0$, one of the roots of this cubic is $\lambda^2=c^2$, the other two being given by the equation

$$x'^2\,(b^2-\lambda^2)+y'^2\,(a^2-\lambda^2)=(a^2-\lambda^2)\,(b^2-\lambda^2),$$

and a root of *this* equation will also be $\lambda^2=c^2$, if

$$\frac{x'^2}{a^2-c^2}+\frac{y'^2}{b^2-c^2}=1.$$

The points on this curve (*the focal ellipse*) therefore belong in a special sense to the value $\lambda^2=c^2$. In like manner the plane $y=0$ separates hyperboloids of one sheet from those of two, and to this limit belongs in a special sense the hyperbola in that plane $\dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2-b^2}=1$, which is called the *focal hyperbola*. The focal conic in the third principal plane is *imaginary*.

[The focal conics represent limiting cases of confocal quadrics. An ellipse may be regarded as a special case either of a hyperboloid of one sheet, or of an ellipsoid. In the former case the generators of the hyperboloid are the tangents to the ellipse and the real points on the "surface" lie *outside* the ellipse; in the latter case the ellipse is regarded as a flattened ellipsoid and the real points are within the ellipse. Strictly speaking the ellipse is the curve of intersection of hyperboloid and ellipsoid when these become "flattened". In like manner a hyperbola may be regarded as representing either a flattened hyperboloid of one sheet, or one of two sheets, or as their curve of intersection. No real conic represents in the same way both an ellipsoid and a hyperboloid of two sheets, since neither of these have real generators. We can see geometrically why two focal conics are real and one imaginary; it depends on the proposition proved in Art. 159.]

159. *The three quadrics which can be drawn through a given point confocal to a given one are respectively an ellipsoid, a hyperboloid of one sheet, and one of two.* For if we substitute in the cubic of the last article successively

$$\lambda^2=a^2,\ \lambda^2=b^2,\ \lambda^2=c^2,\ \lambda^2=-\infty,$$

we get results successively $+\ -\ +\ -$, which prove that the equation has always three real roots, one of which is less than c^2, the second between c^2 and b^2, and the third between b^2 and a^2; and it was shown in the last article that the surfaces corresponding to these values of λ^2 are respectively an ellipsoid, a hyperboloid of one sheet, and one of two.*

* See illustration, fig. 4, facing p. 81.

[159*a*. *The reciprocal, with regard to a sphere, of a system of confocal quadrics is a system of quadrics having the same planes of circular section.* For using Art. 127, the reciprocal system is

$$(xx' + yy' + zz' + k^2)^2 = a^2x^2 + b^2y^2 + c^2z^2 + \lambda^2(x^2 + y^2 + z^2).$$

Moreover the tangential equation of the system of confocals is

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - \delta^2 + \lambda^2(\alpha^2 + \beta^2 + \gamma^2) = 0$$

or $\Sigma + \lambda^2\Omega = 0$ where $\Sigma = 0$ represents any quadric of the system and $\Omega = 0$ represents the circle at infinity. From this the result of Art. 146 may be inferred.

Hence by the method of duality all properties of a confocal system correspond to properties of a concentric system of quadrics having the same planes of circular sections.

Since $\alpha^2 + \beta^2 + \gamma^2$ is an invariant for all rectangular axes, the tangential equation of a system of confocals referred to any rectangular axes is of the form $\Sigma + \lambda^2\Omega = 0$, where $\Sigma = 0$ is the equation of one quadric of the system and λ^2 is proportional to the difference of the squares of the axes of $\Sigma + \lambda^2\Omega = 0$ and $\Sigma = 0$.

Ex. 1. Prove that two confocal quadrics of the same species do not intersect in real points, while those of different species intersect in a real curve.

Ex. 2. Investigate the system of confocal cones

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} + \frac{z^2}{c^2 - \lambda^2} = 0.]$$

160. Another convenient way of solving the problem to describe through a given point quadrics confocal to a given one, is to take for the unknown quantity the primary axis of the sought confocal surface. Then since we are given $a'^2 - b'^2$ and $a'^2 - c'^2$ which we shall call h^2 and k^2, we have the equation

$$\frac{x'^2}{a'^2} + \frac{y'^2}{a'^2 - h^2} + \frac{z'^2}{a'^2 - k^2} = 1,$$

or

$$a'^6 - a'^4\,(h^2 + k^2 + x'^2 + y'^2 + z'^2)$$
$$+ a'^2\{h^2k^2 + x'^2(h^2 + k^2) + y'^2k^2 + z'^2h^2\} - x'^2h^2k^2 = 0.$$

From this equation we can at once *express the coordinates of the intersection of three confocal surfaces in terms of their axes.* Thus if a'^2, a''^2, a'''^2 be the roots of the above equation, the last term of it gives us at once $x'^2h^2k^2 = a'^2a''^2a'''^2$, or

$$x'^2 = \frac{a'^2a''^2a'''^2}{(a^2 - b^2)(a^2 - c^2)}.$$

And by parity of reasoning, since we might have taken b^2 or c^2 for our unknown, we have

$$y'^2 = \frac{b'^2b''^2b'''^2}{(b^2-a^2)(b^2-c^2)},\ z'^2 = \frac{c'^2c''^2c'''^2}{(c^2-a^2)(c^2-b^2)}.*$$

N.B.—In the above we suppose b'^2, b''^2, &c., to involve their signs implicitly. Thus c''^2 belonging to a hyperboloid of one sheet is essentially negative, as are also b'''^2 and c'''^2.

From the last article it follows that if $a' > a'' > a'''$ then a' is the semi axis of the ellipsoid, a'' of the hyperboloid of one sheet, and a''' of the hyperboloid of two sheets. We observe also that k is the lower limit of a' and the upper limit of a'', while h is the lower limit of a'' and the upper of a''', that is in general $$a' > k > a'' > h > a'''.$$

161. The preceding cubic also enables us to express the *radius vector to the point of intersection in terms of the axes.* For the second term of it gives us

$$x'^2 + y'^2 + z'^2 + (a^2 - b^2) + (a^2 - c^2) = a'^2 + a''^2 + a'''^2,$$

or $$x'^2 + y'^2 + z'^2 = a'^2 + b''^2 + c'''^2.$$

162. *Two confocal surfaces cut each other everywhere at right angles.*

Let $x'y'z'$ be any point common to the two surfaces, p' and p'' the lengths of the perpendiculars from the centre on the tangent plane to each at that point, then (Art. 89) the direction-cosines of these two perpendiculars are

$$\frac{p'x'}{a'^2},\ \frac{p'y'}{b'^2},\ \frac{p'z'}{c'^2};\ \frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2}.$$

And the condition that the two should be at right angles is (Art. 13)

$$p'p''\left\{\frac{x'^2}{a'^2a''^2} + \frac{y'^2}{b'^2b''^2} + \frac{z'^2}{c'^2c''^2}\right\} = 0.$$

But since the coordinates $x'y'z'$ satisfy the equations of both surfaces we have

* These expressions show that the umbilics are the points where a quadric is met by its focal conics; e.g. the real umbilics on an ellipsoid are the points where it is met by its focal hyperbola (cf. Art. 149). But for the focal hyperbola $a''^2 = a'''^2 = a^2 - b^2$ and the coordinates are

$$x^2 = a^2\frac{a^2-b^2}{a^2-c^2},\ y = 0,\ z^2 = c^2\frac{b^2-c^2}{a^2-c^2}.$$

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}+\frac{z'^2}{c'^2}=1,\ \frac{x'^2}{a''^2}+\frac{y'^2}{b''^2}+\frac{z'^2}{c''^2}=1.$$

And if we subtract one of these equations from the other and remember that $a''^2-a'^2=b''^2-b'^2=c''^2-c'^2$, the remainder is

$$(a''^2-a'^2)\left\{\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}\right\}=0,$$

which was to be proved.

At the point therefore where three confocals intersect, each tangent plane cuts the other two perpendicularly, and the tangent plane to any one contains the normals to the other two.

163. *If a plane be drawn through the centre parallel to any tangent plane to a quadric, the axes of the section made by that plane are parallel to the normals to the two confocals through the point of contact.*

It has been proved that the parallels to the normals are at right angles to each other, and it only remains to be proved that they are conjugate diameters in their section. But (Art. 94) the condition that two lines should be conjugate diameters is

$$\frac{\cos\alpha\cos\alpha'}{a'^2}+\frac{\cos\beta\cos\beta'}{b'^2}+\frac{\cos\gamma\cos\gamma'}{c'^2}=0.$$

The direction-conics then of the normals being

$$\frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2};\ \frac{p'''x'}{a'''^2},\ \frac{p'''y'}{b'''^2},\ \frac{p'''z'}{c'''^2},$$

we have to prove that

$$p''p'''\left\{\frac{x'^2}{a'^2a''^2a'''^2}+\frac{y'^2}{b'^2b''^2b'''^2}+\frac{z'^2}{c'^2c''^2c'''^2}\right\}=0.$$

But the truth of this equation appears at once on subtracting one from the other the equations which have been proved in the last article,

$$\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}=0,\ \frac{x'^2}{a'^2a'''^2}+\frac{y'^2}{b'^2b'''^2}+\frac{z'^2}{c'^2c'''^2}=0.$$

164. *To find the lengths of the axes of the central section of a quadric by a plane parallel to the tangent plane at the point $x'y'z'$.*

From the equation of the surface the length of a central radius vector whose direction-angles are α, β, γ is given by the equation

$$\frac{1}{\rho^2}=\frac{\cos^2\alpha}{a'^2}+\frac{\cos^2\beta}{b'^2}+\frac{\cos^2\gamma}{c'^2}.$$

Put for α, β, γ the values given in the last article, and we find for the length of one of these axes,

$$\frac{1}{\rho^2}=p''^2\left\{\frac{x'^2}{a'^2a''^4}+\frac{y'^2}{b'^2b''^4}+\frac{z'^2}{c'^2c''^4}\right\}.$$

Now we have the equations,

$$\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}=0,$$

$$\frac{x'^2}{a''^4}+\frac{y'^2}{b''^4}+\frac{z'^2}{c''^4}=\frac{1}{p''^2}.$$

Subtracting we have

$$\frac{x'^2}{a'^2a''^4}+\frac{y'^2}{b'^2b''^4}+\frac{z'^2}{c'^2c''^4}=\frac{1}{p''^2(a'^2-a''^2)}.$$

And substituting this value in the expression already found for ρ^2 we get $\rho^2=a'^2-a''^2$. In like manner the square of the other axis is $a'^2-a'''^2$.

Hence, *if two confocal quadrics intersect, and a radius of one be drawn parallel to the normal to the other at any point of their curve of intersection, this radius is of a constant length.*

165. Since the product of the axes of a central section by the perpendicular on a parallel tangent plane is equal to abc (Art. 96), we get immediately expressions for the lengths p', p'', p'''. We have

$$p'^2=\frac{a'^2b'^2c'^2}{(a'^2-a''^2)\,(a'^2-a'''^2)},\quad p''^2=\frac{a''^2b''^2c''^2}{(a''^2-a'^2)\,(a''^2-a'''^2)},$$

$$p'''^2=\frac{a'''^2b'''^2c'''^2}{(a'''^2-a'^2)\,(a'''^2-a''^2)}.$$

The reader will observe the symmetry which exists between these values for p'^2, p''^2, p'''^2, and the values already found for x'^2, y'^2, z'^2. If the three tangent planes had been taken as coordinate planes, p', p'', p''' would be the coordinates of the centre of the surface. The analogy then between the values

for p', p'', p''', and those for x', y', z', may be stated as follows: *With the point $x'y'z'$ as centre three confocals may be described having the three tangent planes for principal planes and intersecting in the centre of the original system of surfaces. The axes of the new system of confocals are a', a'', a'''; b', b'', b'''; c', c'', c'''. The three tangent planes to the new system are the three principal planes of the original system.*

If a central section through $x'y'z'$ be parallel to one of these principal planes (the plane of yz for instance) in the surface to which this latter is a tangent plane, it appears from Art. 164 that the squares of its axes are a^2-b^2, a^2-c^2. It follows then that the directions and magnitudes of the axes of the section are the same, no matter where the point $x'y'z'$ be situated. The squares of the axes are equal, with signs changed, to the squares of the axes of the corresponding focal conic.

166. If D be the diameter of a quadric parallel to the tangent line at any point of its intersection with a confocal, and p the perpendicular on the tangent plane at that point, then *pD is constant for every point on that curve of intersection.* For the tangent line at any point of the curve of intersection of two surfaces is the intersection of their tangent planes at that point, which in this case (Art. 162) is normal to the third confocal through the point. Hence (Art. 164) $D^2=a'^2-a'''^2$, and therefore (Art. 165) $p^2D^2=\dfrac{a'^2b'^2c'^2}{a'^2-a''^2}$, which is constant if a', a'' be given.

167. *To find the locus of the pole of a given plane with regard to a system of confocal surfaces.*

Let the given plane be $Ax+By+Cz=1$, and its pole $\xi\eta\zeta$; then we must identify the given equation with

$$\frac{x\xi}{a^2-\lambda^2}+\frac{y\eta}{b^2-\lambda^2}+\frac{z\zeta}{c^2-\lambda^2}=1,$$

whence $$\frac{\xi}{a^2-\lambda^2}=A,\ \frac{\eta}{b^2-\lambda^2}=B,\ \frac{\zeta}{c^2-\lambda^2}=C.$$

Eliminating λ^2 between these equations we find, for the equations of the locus,

$$\frac{x}{A} - a^2 = \frac{y}{B} - b^2 = \frac{z}{C} - c^2.$$

The locus is therefore a right line perpendicular to the given plane.

The theorem just proved implicitly contains the solution of the problem, *to describe a surface confocal to a given one to touch a given plane.* For, since the pole of a tangent plane to a surface is its point of contact, it is evident that but one surface can be described to touch the given plane, its point of contact being the point where the locus line just determined meets the plane. The theorem of this article may also be stated—*The locus of the pole of a tangent plane to any quadric, with regard to any confocal, is the normal to the first surface.*

168. *To find an expression for the distance between the point of contact of any tangent plane, and its pole with regard to any confocal surface.*

Let $x'y'z'$ be the point of contact of a tangent plane to the surface whose axes are a, b, c; $\xi\eta\zeta$ the pole of the same plane with regard to the surface whose axes are a', b', c'. Then, as in the last article, we have

$$\frac{x'}{a^2} = \frac{\xi}{a'^2}, \quad \frac{y'}{b^2} = \frac{\eta}{b'^2}, \quad \frac{z'}{c^2} = \frac{\zeta}{c'^2},$$

whence $\quad \xi - x' = \dfrac{a'^2 - a^2}{a^2} x', \quad \eta - y' = \dfrac{b'^2 - b^2}{b^2} y', \quad \zeta - z' = \dfrac{c'^2 - c^2}{c^2} z'.$

Squaring and adding,

$$D^2 = (a'^2 - a^2)^2 \left\{\frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}\right\},$$

whence $D = \dfrac{a'^2 - a^2}{p}$, whence p is the perpendicular from the centre on the plane.

169. *The axes of any tangent cone to a quadric are the normals to the three confocals which can be drawn through the vertex of the cone.*

Consider the tangent plane to one of these three surfaces which pass through the vertex $x'y'z'$; then the pole of that plane with regard to the original surface lies (Art. 65) on the polar plane of $x'y'z'$, and (Art. 167) on the normal to the exterior surface. It is therefore the point where that normal meets the polar plane of $x'y'z'$, that is to say, the plane of contact of the cone.

It follows, then (Art. 64), that the three normals meet this plane of contact in three points, such that each is the pole of the line joining the other two with respect to the section of the surface by that plane. But since this is also a section of the cone, it follows (Art. 71) that the three normals are a system of conjugate diameters of the cone, and since they are mutually at right angles they are its axes.

170. *If at any point on a quadric a line be drawn touching the surface and through that line two tangent planes to any confocal, these two planes will make equal angles with the tangent plane at the given point on the first quadric.* For, by the last article, that tangent plane is a principal plane of the cone touching the confocal surface and having the given point for its vertex, and the two tangent planes will be tangent planes of that cone. But two tangent planes to any cone drawn through a line in a principal plane make equal angles with that plane.

The *focal cones* (that is to say, the cones whose vertices are any points and which stand on the focal conics) are limiting cases of cones enveloping confocal surfaces, and it is still true that the two tangent planes to a focal cone drawn through any tangent line on a surface make equal angles with the tangent plane in which that tangent line lies. If the surface be a cone its focal conic reduces to two right lines, and the theorem just stated in this case becomes, that *any tangent plane to a cone makes equal angles with the planes containing its edge of contact and each of the focal lines.* This theorem, however, will be proved independently in Chap. x.

171. It follows, from Art. 169, that if the three normals be made the axes of coordinates, the equation of the cone must take the form $Ax^2+By^2+Cz^2=0$. To verify this by actual transformation will give us an independent proof of the theorem of Art. 169, and a knowledge of the actual values of A, B, C will be useful to us afterwards.

The equation of the tangent cone given, Art. 78, is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2.$$

If the axes be transformed to parallel axes passing through the vertex of the cone, this equation becomes, as is easily seen,

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}\right)^2.$$

Now to transform to the three normals as axes, we have to substitute the direction-cosines of these lines in the formulæ of Art. 17, and we see that we have to substitute

$$\text{for } x,\ \frac{p'x'}{a'^2}x+\frac{p''x'}{a''^2}y+\frac{p'''x'}{a'''^2}z,$$

$$\text{for } y,\ \frac{p'y'}{b'^2}x+\frac{p''y'}{b''^2}y+\frac{p'''y'}{b'''^2}z,$$

$$\text{for } z,\ \frac{p'z'}{c'^2}x+\frac{p''z'}{c''^2}y+\frac{p'''z'}{c'''^2}z.$$

In order more easily to see the result of this substitution the following preliminary formulæ will be useful.

Let $$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1=S,*$$

then since $$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}+\frac{z'^2}{c'^2}-1=0,$$

we have $$\frac{x'^2}{a^2a'^2}+\frac{y'^2}{b^2b'^2}+\frac{z'^2}{c^2c'^2}=\frac{S}{a'^2-a^2}.$$

In like manner $$\frac{x'^2}{a^2a''^2}+\frac{y'^2}{b^2b''^2}+\frac{z'^2}{c^2c''^2}=\frac{S}{a''^2-a^2}.$$

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* It may be observed that this quantity S is equal to

$$\frac{(a'^2-a^2)\ (a''^2-a^2)\ (a'''^2-a^2)}{a^2b^2c^2},$$

for $a^2-a'^2$, $a^2-a''^2$, $a^2-a'''^2$ are the roots of the cubic of Art. 158, whose absolute term is $a^2b^2c^2S$.

and hence $$\frac{x'^2}{a^2a'^2a''^2}+\frac{y'^2}{b^2b'^2b''^2}+\frac{z'^2}{c^2c'^2c''^2}=\frac{S}{(a'^2-a^2)(a''^2-a^2)}.$$

Lastly, since $\dfrac{x'^2}{a'^4}+\dfrac{y'^2}{b'^4}+\dfrac{z'^2}{c'^4}=\dfrac{1}{p'^2}$,

and $$\frac{x'^2}{a'^2a^2}+\frac{y'^2}{b'^2b^2}+\frac{z'^2}{c'^2c^2}=\frac{S}{a'^2-a^2},$$

we have $$\frac{x'^2}{a'^4a^2}+\frac{y'^2}{b'^4b^2}+\frac{z'^2}{c'^4c^2}=\frac{S}{(a'^2-a^2)^2}-\frac{1}{p'^2(a'^2-a^2)}.$$

When now we make the transformation directed, in the left-hand side of the equation of the tangent cone, the coefficient of x^2 is found to be

$$p'^2S\left\{\frac{x'^2}{a'^4a^2}+\frac{y'^2}{b'^4b^2}+\frac{z'^2}{c'^4c^2}\right\},$$

and that of xy is

$$2p'p''S\left\{\frac{x'^2}{a^2a'^2a''^2}+\frac{y'^2}{b^2b'^2b''^2}+\frac{z'^2}{c^2c'^2c''^2}\right\}.$$

The left-hand side therefore of the transformed equation is

$$S^2\left(\frac{p'x}{a'^2-a^2}+\frac{p''y}{a''^2-a^2}+\frac{p'''z}{a'''^2-a^2}\right)^2$$
$$-S\left\{\frac{x^2}{a'^2-a^2}+\frac{y^2}{a''^2-a^2}+\frac{z^2}{a'''^2-a^2}\right\}.$$

But the quantity $\dfrac{xx'}{a^2}+\dfrac{yy'}{b^2}+\dfrac{zz'}{c^2}$ treated in like manner becomes

$$S\left(\frac{p'x}{a'^2-a^2}+\frac{p''y}{a''^2-a^2}+\frac{p'''z}{a'''^2-a^2}\right).$$

Its square therefore destroys the first group of terms on the other side of the equation, and *the equation of the cone becomes*

$$\frac{x^2}{a'^2-a^2}+\frac{y^2}{a''^2-a^2}+\frac{z^2}{a'''^2-a^2}=0,$$

which is the required transformed equation of the tangent cone.

[172. The equation of the cone referred to its axes can be found with less trouble as follows :—

Let n', n'', n''' be the intercepts made on the axes of the cone between the vertex and the polar plane ($L=0$) of the vertex with regard to the quadric. Then $L\equiv\dfrac{x}{n'}+\dfrac{y}{n''}+\dfrac{z}{n'''}-1$. But L is parallel to the polar plane (M) of the centre G with regard to the cone (since the terms of the highest degree are easily seen to be the same for both equations, when referred to the axes of the

ellipsoid). But $M \equiv Axp' + Byp'' + Czp'''$, since p', p'', p''' are the coordinates of G. Hence

$$An'p' = Bn''p'' = Cn'''p'''.$$

Now by Arts. 168, 169, or directly, we find $n'p' = a'^2 - a^2$ &c.; therefore the equation of the cone is that found in the last article.]

[173. To illustrate the use of tangential coordinates we prove the results of Arts. 169, 171, by using the tangential equation of a system of confocals (Art. 159*a*). Take the normals through a point P (a', a'', a''') for axes and let $\Sigma \equiv (A, B, C, D, F, G, H, L, M, N)\ (\alpha, \beta, \gamma, \delta)^2$ while $\Omega \equiv \alpha^2 + \beta^2 + \gamma^2$. The three confocals through the point are $\Sigma - \lambda'^2\Omega = 0$, $\Sigma - \lambda''^2\Omega = 0$, $\Sigma - \lambda'''^2\Omega = 0$, where λ'^2, λ''^2, λ'''^2, are proportional to $a'^2 - a^2$, &c. The plane $x = 0$, whose plane coordinates are 1, 0, 0, 0, is a tangent plane to $\Sigma - \lambda'^2\Omega = 0$ and the origin, whose equation is $\delta = 0$, is the point of contact. Hence (Art. 129) $A\alpha + H\beta + G\gamma + L\delta - \lambda'^2\alpha = 0$ reduces to $\delta = 0$; therefore $A = \lambda'^2$, $H = 0$, $G = 0$. Similarly $B = \lambda''^2$, $C = \lambda'''^2$, $F = 0$. Thus the tangential equation of the quadric is

$$\Sigma \equiv \lambda'^2\alpha^2 + \lambda''^2\beta^2 + \lambda'''^2\gamma^2 + 2L\alpha\delta + 2M\beta\delta + 2N\gamma\delta + D\delta^2 = 0.$$

But the tangential equations of the cone from the origin are the above and $\delta = 0$ and are therefore $\lambda'^2\alpha^2 + \lambda''^2\beta^2 + \lambda'''^2\gamma^2 = \delta = 0$.

Returning to point coordinates, the equation of the cone is thus

$$\frac{x^2}{\lambda'^2} + \frac{y^2}{\lambda''^2} + \frac{z^2}{\lambda'''^2} = 0.$$

The equation $\Sigma = 0$ may be reduced to the form

$$(a'^2 - a^2)\,\alpha^2 + (a''^2 - a^2)\,\beta^2 + (a'''^2 - a^2)\,\gamma^2 + 2\,(p'\alpha + p''\beta + p'''\gamma)\,\delta + \delta^2 = 0.$$

This is proved by expressing, first, the condition that the tangential equation of the centre G is $\alpha p' + \beta p'' + \gamma p''' + \delta = 0$, and that it is also the pole of the plane at infinity, whose coordinates are 0, 0, 0, 1; and, secondly, the condition that the coordinates of the polar plane of the origin (Art. 172) are $\frac{1}{n'}$, $\frac{1}{n''}$, $\frac{1}{n'''}$, -1.]

174. As a particular case of the preceding may be found *the equation of either focal cone* (Art. 170); that is to say, the cone whose vertex is any point $x'y'z'$ and which stands on the focal ellipse or focal hyperbola. These answer to the values $a^2 - c^2$, $a^2 - b^2$ for the square of the primary axis: the equations therefore are

$$\frac{x^2}{c'^2} + \frac{y^2}{c''^2} + \frac{z^2}{c'''^2} = 0,$$

$$\frac{x^2}{b'^2} + \frac{y^2}{b''^2} + \frac{z^2}{b'''^2} = 0.$$

These equations might also have been found, by forming, as

in Ex. 9, Art. 121, the equations of the focal cones, and then transforming them as in the last articles.

It may be seen without difficulty that any normal and the corresponding tangent plane meet any of the principal planes in a point and line which are pole and polar with regard to the focal conic in that plane. This is a particular case of Art. 169.

The formulæ employed in Art. 171 enable us to transform to the same new axes any other equations.

Ex. 1. To transform the equation of the quadric itself to the three normals through any point $x'y'z'$ as axes. The equation transformed to parallel axes becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + S + 2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right) = 0.$$

And when the axes are turned round, we get

$$S\left(\frac{p'x}{a'^2 - a^2} + \frac{p''y}{a''^2 - a^2} + \frac{p'''z}{a'''^2 - a^2} + 1\right)^2 = \frac{x^2}{a'^2 - a^2} + \frac{y^2}{a''^2 - a^2} + \frac{z^2}{a'''^2 - a^2}.$$

The quantity under the brackets on the left-hand side of the equation is evidently the transformed equation of the polar plane of the point.

This may also be deduced from the equation $\Sigma = 0$ of Art. 173.

Ex. 2. The preceding equation is somewhat modified if the point $x'y'z'$ is on the surface. The equation transformed to parallel axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right) = 0.$$

Let now
$$p^2\left\{\frac{x'^2}{a^6} + \frac{y'^2}{b^6} + \frac{z'^2}{c^6}\right\} = \frac{1}{\gamma^2};$$

then the equation, transformed to the three normals as axes is

$$\frac{x^2}{\gamma^2} + \frac{y^2}{a^2 - a'^2} + \frac{z^2}{a^2 - a''^2} - \frac{2p'xy}{p(a^2 - a'^2)} - \frac{2p''xz}{p(a^2 - a''^2)} + \frac{2x}{p} = 0.$$

It is observed that γ is the diameter parallel to the normal at the point $x'y'z'$, and that we have

$$\frac{1}{\gamma^2} + \frac{1}{a^2 - a'^2} + \frac{1}{a^2 - a''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2};$$

and the transformed equation may be otherwise written

$$\frac{(p'x - py)^2}{a^2 - a'^2} + \frac{(p''x - pz)^2}{a^2 - a''^2} + (x + p)^2 = p^2.$$

Ex. 3. To transform the equation of the reciprocal surface with regard to any point to the three normals through the point. The equation is (Art. 127)

$$(xx' + yy' + zz' + k^2)^2 = a^2x^2 + b^2y^2 + c^2z^2,$$

and the transformed equation is found to be

$$(a'^2 - a^2)x^2 + (a''^2 - a^2)y^2 + (a'''^2 - a^2)z^2 + 2k^2(p'x + p''y + p'''z) + k^4 = 0.$$

This equation is evidently another form of the equation $\Sigma = 0$ of Art. 173.

175. To return to the equation of the tangent cone (Art. 171). Its form proves that *all cones having a common vertex and circumscribing a series of confocal surfaces are coaxal and confocal.* For the three normals through the common vertex are axes to every one of the system of cones; and the form of the equation shows that the differences of the squares of the axes are independent of a^2. The *equations of the common focal lines of the cones* are (Art. 151)

$$\frac{x^2}{a'^2 - a''^2} = \frac{z^2}{a''^2 - a'''^2}; \; y^2 = 0.$$

But it was proved (Art. 164) that the central section of the hyperboloid of one sheet which passes through $x'y'z'$ is

$$\frac{x^2}{a''^2 - a'^2} + \frac{z^2}{a''^2 - a'''^2} = 1,$$

and the section of the hyperboloid by the tangent plane itself is similar to this, or is also

$$\frac{x^2}{a'^2 - a''^2} - \frac{z^2}{a''^2 - a'''^2} = 0.$$

Hence the *focal lines of the system of cones are the generating lines of the hyperboloid which passes through the point*—a theorem due to Chasles, *Liouville*, XI. p. 121, and also noticed by Jacobi (*Crelle*, Vol. XII. p. 137).

This may also be proved thus: Take any edge of one of the system of cones, and through it draw a tangent plane to that cone and also planes containing the generating lines of the hyperboloid: these latter planes are tangent planes to the hyperboloid, and therefore (Art. 170) make equal angles with the tangent plane to the cone. The two generators are therefore such that the planes drawn through them and through any edge of the cone make equal angles with the tangent plane to the cone; but this is a property of the focal lines (Art. 170).

COR. *If a system of confocals be projected orthogonally on any plane, the projections of the contours are confocal conics.* The projections are the sections by that plane of cylinders perpendicular to it, and enveloping the quadrics. And these cylinders may be considered as a system of enveloping cones

whose vertex is the point at infinity on the common direction of their generators.

176. *Two confocal surfaces can be drawn to touch a given line.*

Take on the line any point $x'y'z'$; let the axes of the three surfaces passing through it be a', a'', a''', and the angles the line makes with the three normals α, β, γ. Then it appears, from Art. 173, that a is determined by the *quadratic*

$$\frac{\cos^2\alpha}{a'^2-a^2}+\frac{\cos^2\beta}{a''^2-a^2}+\frac{\cos^2\gamma}{a'''^2-a^2}=0.$$

If a and a' be the roots of this quadratic, the two cones

$$\frac{x^2}{a'^2-\mathrm{a}^2}+\frac{y^2}{a''^2-\mathrm{a}^2}+\frac{z^2}{a'''^2-\mathrm{a}^2}=0,\quad \frac{x^2}{a'^2-\mathrm{a}'^2}+\frac{y^2}{a''^2-\mathrm{a}'^2}+\frac{z^2}{a'''^2-\mathrm{a}'^2}=0$$

have the given line as a common edge, and it is proved, precisely as at Art. 162, that the tangent planes to the cones through this line are at right angles to each other. And since the tangent planes to a tangent cone to a surface, by definition touch that surface, it follows that *the tangent planes drawn through any right line to the two confocals which it touches are at right angles to each other.*

The property that the tangent cones from any point to two intersecting confocals cut each other at right angles is sometimes expressed as follows: *the contours of two confocals seen from any point appear to intersect everywhere at right angles.* [They 'appear' to intersect at four points corresponding to the four common generators of the tangent cones].

[176*a*. *To find the conditions that the tangent cone from a point a_1, a_2, a_3 (where $a_1 > a_2 > a_3$) to a confocal whose major axis is α, may be real.*

(1) If α is an ellipsoid, $\alpha > a_2$, and the condition is $\alpha < a_1$, for then the cone is of the form $Ax^2 - By^2 - Cz^2 = 0$, where A, B, C are positive.

(2) If α is a hyperboloid of one sheet, $\alpha < a_1$ and $\alpha > a_3$ and the tangent cone is always real, being of the form $Ax^2 \pm By^2 - Cz^2 = 0$.

(3) If α is a hyperboloid of two sheets, $\alpha < a_2$, and the condition is $\alpha > a_3$, the cone being of the form $Ax^2 + By^2 - Cz^2 = 0$.

Next, *to find the condition that the (real) tangent cones from (a_1, a_2, a_3) to two confocals (α, α') may intersect in real generators:*

It is easy to see, by subtracting the equation of one cone from that of the other, that if the confocal cones

$$lx^2 + my^2 + nz^2 = 0,\ l'x^2 + m'y^2 + n'z^2 = 0$$

intersect in real generators then ll', mm', nn' cannot all have the same sign. Assuming $\alpha > \alpha'$, the cones must therefore be of the forms

$$Ax^2 - By^2 - Cz^2 = 0,\ A'x^2 + B'y^2 - C'z^2 = 0,$$

the coefficients being positive; and solving for $x^2 : y^2 : z^2$, we see that, if these conditions are fulfilled the cones intersect in four real generators. The conditions are therefore $\alpha > a_2$, and $\alpha' < a_2$. Hence one of the quadrics is either an ellipsoid or a hyperboloid of one sheet, and the other is either a hyperboloid of one sheet or a hyperboloid of two sheets.

Real tangent cones from a point to two intersecting confocals need not intersect in real generators, or, in other words, the quadrics may intersect without appearing to do so. This will happen if the numbers a_1, α, k, α', a_2 are in descending order, or if the numbers a_2, α, h, α', a_3 are in descending order. The first case may be illustrated by imagining the appearance which the ellipsoid and the hyperboloid of one sheet in fig. 4, p. 81, would present if seen from a distant point on the vertical axis.

Again, two confocals will appear to intersect without really doing so if the numbers k, α, a_2, α', h are in descending order; that is, if two hyperboloids of one sheet are viewed from a point which lies between them.

Ex. If P be a point from which real tangent cones can be drawn to three intersecting confocals, U, V, W (U being an ellipsoid, V and W hyperboloids of one sheet and of two sheets respectively) prove (1) that V (seen from P) cannot appear to intersect both U and W; and find the conditions, (2) that U may appear to intersect both V and W, and (3) that W may appear to intersect both U and V.]

177. *If through a given line tangent planes be drawn to a system of confocals, the corresponding normals generate a hyperbolic paraboloid.*

The normals are evidently parallel to one plane; namely, the plane perpendicular to the given line; and if we consider any one of the confocals, then, by Art. 167, the normal to any plane through the line contains the pole of that plane with regard to the assumed confocal, which pole is a point on the polar line of the given line with regard to that confocal. Hence, every normal meets the polar line of the given line with regard to any confocal. The surface generated by the normals is therefore a hyperbolic paraboloid (Art. 116). It is evident that the surface generated by the polar lines, just

referred to, is the same paraboloid, of which they form the other system of generators.

The points in which this paraboloid meets the given line are the two points where this line touches confocals.

A special case occurs when the given line is itself a normal to a surface U of the system. The normal corresponding to any plane drawn through that line is found by letting fall a perpendicular on that plane from the pole of the same plane with regard to U (Art. 167), but it is evident that both pole and perpendicular must lie in the tangent plane to U to which the given line is normal. Hence, in this case all the normals lie in the same plane.

From the principle that the anharmonic ratio of four planes passing through a line is the same as that of their four poles with regard to any quadric, it is found at once that any four of the normals divide homographically all the polar lines corresponding to the given line with respect to the system of surfaces. In the special case now under consideration, the normals will therefore envelope a conic, which conic will be a parabola, since the normal in one of its positions may lie at infinity; namely, when the surface is an infinite sphere (Art. 158). The point where the given line meets the surface to which it is normal lies on the directrix of this parabola.

178. If α, β, γ be the direction-angles, referred to the three normals through the vertex, of the perpendicular to a tangent plane of the cone of Arts. 171, &c., since this perpendicular lies on the reciprocal cone, α, β, γ must satisfy the relation.

$$(a'^2 - a^2)\cos^2\alpha + (a''^2 - a^2)\cos^2\beta + (a'''^2 - a^2)\cos^2\gamma = 0,$$

or
$$a'^2\cos^2\alpha + a''^2\cos^2\beta + a'''^2\cos^2\gamma = a^2.$$

This relation enables us at once *to determine the axis of the surface which touches any plane*, for if we take any point on the plane, we know a', a'', a''' for that point, as also the angles which the three normals through the point make with the plane, and therefore a^2 is known.

179. If the relation of the last article were proved independently, we should, by reversing the steps of the demon-

stration, obtain a proof without transformation of coordinates of the equation of the tangent cone (Art. 171). The following proof is due to Chasles: The quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is the sum of the squares of the projections on a perpendicular to the given plane of the lines a', a'', a'''. We have seen (Art. 165) that these are the axes of a surface having $x'y'z'$ for its centre and passing through the original centre. And it was proved in the same article that three other conjugate diameters of the same surface are the radius vector from the centre to $x'y'z'$, together with two lines parallel to two of the original axes whose squares are $a^2 - b^2$, $a^2 - c^2$. It was also proved (Art. 98) that the sum of the squares of the projections on any line of three conjugate diameters of a quadric is equal to that of any other three conjugate diameters. It follows then that the quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is equal to the sum of the squares of the projections on the perpendicular from the centre on the given plane, of the radius vector, and of two lines whose magnitude and direction are known. The projections of the last two lines are constant, while the projection of the radius vector is the perpendicular itself which is constant if $x'y'z'$ belong to the given plane. It is proved then that the quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is constant while the point $x'y'z'$ moves in a given plane; and it is evident that the constant value is the a^2 of the surface which touches the given plane, since for it we have

$$\cos\alpha = 1,\ \cos\beta = 0,\ \cos\gamma = 0.$$

180. *The locus of the intersection of three planes mutually at right angles, each of which touches one of three confocals is a sphere.*

This is proved as in Art. 93.

Add together

$$p^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma,$$
$$p'^2 = a'^2 \cos^2\alpha' + b'^2 \cos^2\beta' + c'^2 \cos^2\gamma',$$
$$p''^2 = a''^2 \cos^2\alpha'' + b''^2 \cos^2\beta'' + c''^2 \cos^2\gamma'',$$

when we get $\rho^2 = a^2 + b^2 + c^2 + (a'^2 - a^2) + (a''^2 - a^2)$, where ρ is the distance from the centre of the intersection of the planes.

Again, by subtracting one from the other, the two equations

$p^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma, p'^2 = a'^2\cos^2\alpha + b'^2\cos^2\beta + c'^2\cos^2\gamma,$

we learn that *the difference of the squares of the perpendiculars on two parallel tangent planes to two confocals is constant and equal* $a^2 - a'^2$.

It may be remarked that the reciprocal of the theorem of Art. 93 is that if from any point O there be drawn three radii vectors to a quadric, mutually at right angles, the plane joining their extremities envelopes a surface of revolution. If O be on the quadric, the plane passes through a fixed point.

181. *Two cones having a common vertex envelope two confocals; to find the length of the intercept made on one of their common edges by a plane through the centre parallel to the tangent plane to a confocal through the vertex.* The intercepts made on the four common edges are of course all equal, since the edges are equally inclined to the plane of section, which is parallel to a common principal plane of both cones.

Let there be any two confocal cones

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 0, \quad \frac{x^2}{\alpha'^2} + \frac{y^2}{\beta'^2} + \frac{z^2}{\gamma'^2} = 0,$$

then for their intersection, we have

$$\frac{x^2}{\alpha^2\alpha'^2(\beta^2 - \gamma^2)} = \frac{y^2}{\beta^2\beta'^2(\gamma^2 - \alpha^2)} = \frac{z^2}{\gamma^2\gamma'^2(\alpha^2 - \beta^2)},$$

and if the common value of these be λ^2, we have

$$x^2 + y^2 + z^2 = \lambda^2(\alpha^2 - \beta^2)(\beta^2 - \gamma^2)(\alpha^2 - \gamma^2).$$

Putting in the values of α^2, β^2, γ^2 from the equations of the

tangent cones (Art. 176), and determining λ^2 by the equation (see Art. 165) $x^2 = \frac{a'^2b'^2c'^2}{(a'^2 - a''^2)(a'^2 - a'''^2)}$, we get for the square of the required intercept

$$\frac{a'^2b'^2c'^2}{(a'^2 - \text{a}^2)(a'^2 - \text{a}'^2)}.$$

If then the confocals be all of different kinds they meet in a point, and this value shows that the intercept is equal to the perpendicular from the centre on the tangent plane to (a', b', c') at their intersection (Art. 165).

In the particular case where the two cones considered are the cones standing on the focal ellipse, and on the focal hyperbola, we have $\text{a}^2 = a^2 - c^2$, $\text{a}'^2 = a^2 - b^2$, and the intercept reduces to a'. Hence, *if through any point on an ellipsoid be drawn a chord meeting both focal conics, the intercept on this chord by a plane through the centre parallel to the tangent plane at the point will be equal to the semi-axis-major of the surface.* This theorem, due to MacCullagh, is analogous to the theorem for plane curves, that a line through the centre parallel to a tangent to an ellipse cuts off on the focal radii portions equal to the semi-axis-major.

182. Chasles has used the principles just established to solve the problem *to determine the magnitude and direction of the axes of a central quadric being given a system of three conjugate diameters.*

Consider first the plane of any two of the conjugate diameters, and we can by plane geometry determine in magnitude and direction the axes of the section by that plane. The tangent plane at P, the extremity of the remaining diameter, will be parallel to the same plane. Now the centre of the given quadric is the point of intersection of three confocals determined as in Art. 165, having the point P for their centre. If now we could construct the focal conics of this new system of confocals, then the two focal cones, whose common vertex is the centre of the original quadric, determine by their mutual intersection four right lines. The six

planes containing these four right lines intersect two by two in the directions of the required axes, while (Art. 181) planes through the point P parallel to the principal planes, cut off on any one of these four lines parts equal in length to the semi-axes.

The focal conics required are immediately constructed. We know the planes in which they lie and the directions of their axes. The squares of their semi-axes are to be $a^2 - a''^2$, $a'^2 - a''^2$; $a^2 - a'^2$, $a'^2 - a''^2$. But now the squares of the semi-axes of the given section are $a^2 - a'^2$, $a^2 - a''^2$ (Art. 164), and these latter axes being known, the axes of the focal conics are immediately found.

183. If through any point P on a quadric a chord be drawn, as in Art. 181, touching two confocals, we can find an expression for the length of that chord. Draw a parallel semi-diameter through the centre, the length of which we shall call R. Now if through P there be drawn a plane conjugate to this diameter, and a tangent plane, they will intercept (counting from the centre) portions on the diameter whose product $= R^2$. But the portion intercepted by the conjugate plane is half the chord required, and the portion intercepted by the tangent plane is the intercept found (Art. 181). Hence

$$C = \frac{2R^2\sqrt{\{(a'^2 - \text{a}^2)\ (a'^2 - \text{a}'^2)\}}}{a'b'c'}.$$

When the chord is that which meets the two focal conics; $\text{a}^2 = a'^2 - c'^2$, $\text{a}'^2 = a'^2 - b'^2$, and $C = \dfrac{2R^2}{a'}$.

184. *To find the locus of the vertices of right cones which can envelope a given surface.*

In order that the equation $\dfrac{x^2}{a'^2 - a^2} + \dfrac{y^2}{a''^2 - a^2} + \dfrac{z^2}{a'''^2 - a^2} = 0$ may represent a right cone, two of the coefficients must be equal; that is to say, $a'' = a'$, or $a'' = a'''$, or in other words, for the point $x'y'z'$ the equation of Art. 158 must have two equal roots. But from what was proved as to the limits within which the roots lie, it is evident that we cannot have equal

roots except when λ is equal to one of the principal semi-axes, i.e. when $x'y'z'$ is on one of the focal conics. This agrees with what was proved (Art. 155).

It appears, hence, as has been already remarked, that the *reciprocal of a surface, with regard to a point on a focal conic, is a surface of revolution*; and *that the reciprocal, with regard to an umbilic, is a paraboloid of revolution.* For an umbilic is a point on a focal conic (Art. 149), and since it is on the surface the reciprocal with regard to it is a paraboloid.

Another particular case of this theorem is, that *two right cylinders can be circumscribed to a central quadric*, the edges of the cylinders being parallel to the asymptotes of the focal hyperbola. For a cone whose vertex is at infinity is a cylinder.

As a particular case of the theorem of this article, the cone standing on the focal ellipse will be a right cone only when its vertex is on the focal hyperbola, and *vice versâ.* This theorem of course may be stated without any reference to the quadrics of which the two conics are focal conics; that *the locus of the vertices of right cones which stand on a given conic is a conic of opposite species in a perpendicular plane.* If the equation of one conic be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, that of the other will be $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2}=1$.

It was proved (Ex. 8, Art. 126) that if a quadric circumscribe a surface of revolution, the cone enveloping the former whose vertex is a focus of the latter is of revolution. From this article then we see that the focal conics of a quadric are the locus of the foci of all possible surfaces of revolution which can circumscribe that quadric.

185. It appears from what has been already said that the focal ellipse and hyperbola are in planes at right angles to each other, and such that the vertices of each coincides with the foci of the other. Two conics so related are each (so to speak) a *locus of foci of the other;* viz. any pair of fixed points F, G on the one conic may be regarded as foci of the

other, the sum or difference of the distances FP, GP to a variable point P on the other, being constant.

Taking the equations of the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2}=1,$$

and introducing the parameters θ, ϕ, as at *Conics*, Arts. 229, 232, the coordinates of a point on each conic may be expressed,

$$a\cos\theta,\ b\sin\theta,\ 0;\ \sec\phi\sqrt{(a^2-b^2)},\ 0,\ b\tan\phi;$$

and the square of the distance between these points is

$$a^2\cos^2\theta-2a\cos\theta\sec\phi\sqrt{(a^2-b^2)}+(a^2-b^2)\sec^2\phi$$
$$+b^2\sin^2\theta+b^2\tan^2\phi,$$

or $a^2\sec^2\phi-2a\cos\theta\sec\phi\sqrt{(a^2-b^2)}+(a^2-b^2)\cos^2\theta$

$$=\{a\sec\phi-\cos\theta\sqrt{(a^2-b^2)}\}^2.$$

And, plainly, the sum or difference of two distances

$$\pm\{a\sec\phi-\cos\theta\sqrt{(a^2-b^2)}\},\ \pm\{a\sec\phi-\cos\theta'\sqrt{(a^2-b^2)}\}$$

is independent of ϕ; and of two distances

$$\pm\{a\sec\phi-\cos\theta\sqrt{(a^2-b^2)}\},\ \pm\{a\sec\phi'-\cos\theta\sqrt{(a^2-b^2)}\}$$

is independent of θ.

Attending to the signs the theorem is this, that if we take two fixed points F, G on the ellipse, the difference $FP-GP$ is constant, being $=+a$ when P is a point on one branch of the hyperbola, and $-a$ when P is on the other. In particular, when F, G are the vertices of the ellipse we have the ordinary focal property of the hyperbola. Again, taking F, G two points on different branches of the hyperbola, the sum $FP+GP$ is constant, and when F, G are the vertices of the hyperbola we have the ordinary focal property of the ellipse. If F, G be taken instead on the same branch of the hyperbola, it is the difference between FP and GP which is constant; and if F and G coincide at a vertex, we have merely the identity $FP-FP=0$, and not a new property of the ellipse *in plano*.

186. The following examples will serve further to illustrate the principles which have been laid down:—

Ex. 1. To find the locus of the intersection of generators to a hyperboloid which cut at right angles.

The section parallel to the tangent plane which contains the generators must be an equilateral hyperbola, so that (Art. 164), $(a''^2 - a'^2) + (a''^2 - a'''^2) = 0$. But (Art. 161) the square of the radius vector to the point is

$$a''^2 + b''^2 + c''^2 - (a''^2 - a'^2) - (a''^2 - a'''^2).$$

We have, therefore, the locus a sphere, the square of whose radius is equal to $a''^2 + b''^2 + c''^2$. Otherwise thus: If two generators are at right angles, their plane together with the plane of each and of the normal at the point, are a system of three tangent planes to the surface, mutually at right angles, whose intersection lies on the sphere $r^2 = a''^2 + b''^2 + c''^2$ (Art. 93).

Ex. 2. To find the locus of the intersection of three tangent lines to a quadric mutually at right angles (see Ex. 6, Art. 121).

Let α, β, γ be the angles made by one of these tangents with the normals through the locus point, and since each of these tangents lies in the tangent cone through that point, we have the conditions

$$\frac{\cos^2\alpha}{a'^2 - a^2} + \frac{\cos^2\beta}{a''^2 - a^2} + \frac{\cos^2\gamma}{a'''^2 - a^2} = 0,$$

$$\frac{\cos^2\alpha'}{a'^2 - a^2} + \frac{\cos^2\beta'}{a''^2 - a^2} + \frac{\cos^2\gamma'}{a'''^2 - a^2} = 0,$$

$$\frac{\cos^2\alpha''}{a'^2 - a^2} + \frac{\cos^2\beta''}{a''^2 - a^2} + \frac{\cos^2\gamma''}{a'''^2 - a^2} = 0.$$

Adding, we have $\dfrac{1}{a'^2 - a^2} + \dfrac{1}{a''^2 - a^2} + \dfrac{1}{a'''^2 - a^2} = 0.$

But $a^2 - a'^2$, $a^2 - a''^2$, $a^2 - a'''^2$ are the three roots of the cubic of Art. 158 which arranged in terms of λ^2 is

$$\lambda^6 + \lambda^4(x^2 + y^2 + z^2 - a^2 - b^2 - c^2) - \lambda^2\{(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2$$
$$- b^2c^2 - c^2a^2 - a^2b^2\} + b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0.$$

And the sum of the reciprocals of the roots will vanish when the coefficient of $\lambda^2 = 0$. This, therefore, gives us the equation of the locus required.

Ex. 3. The section of an ellipsoid by the tangent plane to the asymptotic cone of a confocal hyperboloid is of constant area.

The area (Art. 96) is inversely proportional to the perpendicular on a parallel tangent plane, and we have

$$p^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma.$$

But since the perpendicular is an edge of the cone reciprocal to the asymptotic cone of the hyperboloid we have

$$0 = a'^2 \cos^2\alpha + b'^2 \cos^2\beta + c'^2 \cos^2\gamma,$$

whence $$p^2 = a^2 - a'^2.$$

Ex. 4. To find the length of the perpendicular from the centre on the polar plane of $x'y'z'$ in terms of the axes of the confocals which pass through that point.

Ans. If $a'^2 - a^2 = h^2$, $a''^2 - a^2 = k^2$, $a'''^2 - a^2 = l^2$,

$$\frac{1}{p^2} = \frac{h^2k^2l^2}{a^2b^2c^2}\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{h^2} + \frac{1}{k^2} + \frac{1}{l^2}\right\}.$$

187. Two points, one on each of two confocal ellipsoids, are said to correspond if

$$\frac{x}{a}=\frac{X}{A},\ \frac{y}{b}=\frac{Y}{B},\ \frac{z}{c}=\frac{Z}{C}.$$

It is evident that *the intersection of two confocal hyperboloids pierces a system of ellipsoids in corresponding points ;* for from the value (Art. 160) $x^2=\frac{a^2a'^2a''^2}{(a^2-b^2)\,(a^2-c^2)}$, the quantity $\frac{x^2}{a^2}$ is constant as long as the hyperboloids, having a'^2, a''^2 for axes, are constant.

It will be observed that, the principal planes being limits of confocal surfaces, points on the principal planes determined by equations of the form $\frac{x'^2}{a^2}=\frac{X^2}{a^2-c^2}$, $\frac{y'^2}{b^2}=\frac{Y^2}{b^2-c^2}$, $Z=0$, correspond to any point $x'y'z'$ on a surface, and *when $x'y'z'$ is in the principal plane, the corresponding point is on the focal conic.*

188. *The points on the plane of xy, which correspond to the intersection of an ellipsoid with a series of confocal surfaces, form a series of confocal conics,* of which the points corresponding to the umbilics are the common foci.

Eliminating z^2 between the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$

we find
$$\frac{(a^2-c^2)x^2}{a^2a'^2}+\frac{(b^2-c^2)y^2}{b^2b'^2}=1,$$

whence the corresponding points are connected by the relation

$$\frac{X^2}{a'^2}+\frac{Y^2}{b'^2}=1.$$

This is evidently an ellipse for the intersections with hyperboloids of one sheet, and a hyperbola for the intersections with hyperboloids of two.

The coordinates of the umbilics are

$$x^2=a^2\frac{a^2-b^2}{a^2-c^2},\ y^2=0,$$

the points corresponding to which are

$$X^2=a^2-b^2,\ Y=0,$$

which are therefore the foci of the system of confocal conics.

Curves on the ellipsoid are sometimes expressed by what are called *elliptic coordinates*; that is to say, by an equation of the form $\phi\ (a', a'')=0$, expressing a relation between the axes of the confocal hyperboloids which can be drawn through the point. Now since it appears from this article that a' is half the sum and a'' half the difference of the distances of the points corresponding to the points of the locus from the points which correspond to the umbilics, we can from the equation $\phi\ (a', a'')=0$ obtain an equation $\phi\ (\rho+\rho', \rho-\rho')=0$, from which we can form the equation of the curve on the principal plane which corresponds to the given locus.

189. *If the intersection of a sphere and a concentric ellipsoid be projected on either plane of circular section by lines parallel to the least (or greatest) axis, the projection will be a circle.*

This theorem is only a particular case of the following: "if any two quadrics have common planes of circular section, any quadric through their intersection will have the same"; a theorem which is evident, since if by making $z=0$ in U and in V, the result in each case represents a circle, making $z=0$ in $U+kV$, must also represent a circle.

It will be useful, however, to investigate this particular theorem directly. If we take as axes the axis of y which is a line in the plane of circular section and a perpendicular to it in that plane, the y will remain unaltered, and the new $x^2=$ the old x^2+z^2. But since by the equation of the plane of circular section $z^2=\frac{c^2}{a^2}\cdot\frac{a^2-b^2}{b^2-c^2}x^2$, the new $x^2=\frac{b^2}{a^2}\cdot\frac{a^2-c^2}{b^2-c^2}x^2$.

But for the intersection of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ x^2+y^2+z^2=r^2,$$

we have
$$\frac{a^2-c^2}{a^2}x^2+\frac{b^2-c^2}{b^2}y^2=r^2-c^2,$$

which, on substituting for x^2 the value $\frac{b^2-c^2}{a^2-c^2}\cdot\frac{a^2}{b^2}x$

becomes
$$\frac{b^2-c^2}{b^2}(x^2+y^2)=r^2-c^2.$$

It will be observed that to obtain the projection on the planes of circular sections we left y unaltered, and substituted for x^2, $\frac{b^2 - c^2}{a^2 - c^2} \cdot \frac{a^2}{b^2} x^2$. But to obtain the points corresponding to any point, as in Art. 187, we substitute for x^2, $\frac{a^2}{a^2 - c^2} x^2$, and for y^2, $\frac{b^2}{b^2 - c^2} y^2$. Now the squares of the former co-ordinates have to those of the latter the constant ratio $(b^2 - c^2) : b^2$. Hence we may immediately infer from the last article that *the projection of the intersection of two confocal quadrics on a plane of circular section of one of them is a conic whose foci are the similar projections of the umbilics;* and, again, that given any curve ϕ (a', a'') on the ellipsoid we can obtain the algebraic equation of the projection of that curve on the plane of circular section.

190. *The distance between two points, one on each of two confocal ellipsoids, is equal to the distance between the two corresponding points.*

We have

$$(x - X)^2 + (y - Y)^2 + (z - Z)^2$$
$$= x^2 + y^2 + z^2 + X^2 + Y^2 + Z^2 - 2(xX + yY + zZ).$$

Now (Art. 161)

$$x^2 + y^2 + z^2 = a^2 + b'^2 + c''^2, \quad X^2 + Y^2 + Z^2 = A^2 + B'^2 + C''^2.$$

But for the corresponding points

$$X'^2 + Y'^2 + Z'^2 = A^2 + b'^2 + c''^2, \quad x'^2 + y'^2 + z'^2 = a^2 + B'^2 + C''^2.$$

The sum of the squares therefore of the central radii to the two points is the same as that for the two corresponding points. But the quantities xX, yY, zZ are evidently respectively equal to $x'X'$, $y'Y'$, $z'Z'$, since $aX' = Ax$, $Ax' = aX$, &c. The theorem of this article, due to Ivory, is of use in the theory of attractions.

Ex. If P_1, P_2 be points on a generator of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and P_1', P_2', corresponding points on the confocal $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - \frac{z^2}{c'^2} = 1$, then $P_1 P_2 = P'_1 P'_2$ and $P'_1 P'_2$ is a generator of the second hyperboloid.

Hence *if a hyperboloid, constructed of a network of threads forming the generators, the threads representing generators of opposite system being fastened at their points of intersection, be deformed so that the threads are kept stretched, the deformed surface will be congruent with* (*i.e. superposable on*) *a confocal hyperboloid.*]

191. *Jacobi's method of constructing confocal ellipsoids.* In order to obtain a property of quadrics analogous to the property of conics that the sum of the focal distances is constant, Jacobi states the latter property as follows: Take the two points C and C' on the ellipse at the extremity of the axis-major, then the same relation $\rho + \rho' = 2a$ which connects the distances from C and C' of any point on the line joining these points, connects also the distances from the foci of any point on the ellipse. Now, in like manner, if we take on the principal section of an ellipsoid the three points (A, B, C) which correspond in the sense explained (Art. 187) to any three points (a, b, c) on the focal ellipse, the same relation which connects the distances from the former points of any point (D) in their plane will also connect the distances from the latter points of any point (d) on the surface. In fact, by Art. 190, the distances of the points on the focal ellipse from a point on the surface will be equal to the distances of the point on the principal plane which *corresponds* to the point on the surface, from the three points in the principal section.*

* Townsend has shown from geometrical considerations (*Cambridge and Dublin Mathematical Journal*, Vol. III. p. 154) that this property only belongs to points on the *modular* focal conics, and in fact the points in the plane y which correspond to any point $x'y'z'$ on an ellipsoid are imaginary, as easily appears from the formula of Art. 189.

Townsend easily *derives Jacobi's mode of generation from MacCullagh's modular property.* For if through any point on the surface we draw a plane parallel to a circular section, it will cut the directrices corresponding to the three fixed foci in a triangle of invariable magnitude and figure, and the distances of the point on the surface from the three foci will be in a constant ratio to its distances from the vertices of this triangle. And a similar triangle can be formed with its sides increased or diminished in a fixed ratio, the distances from the vertices of which to the point $x'y'z'$ shall be equal to its distances from the foci.

[We can thus construct geometrically a system of confocal ellipsoids by means of a system of ellipses confocal with a given ellipse F. Let S be an ellipse external to F and confocal thereto. Take any three points a, b, c on F and let A, B, C be the corresponding points on S. Take any point D in the plane inside F, and construct a pyramid whose vertices are a, b, c, d where $ad = AD$, $bd = BD$, $cd = CD$. Then the locus of d as D moves in the plane is an ellipsoid of which S is a principal section, and by varying S we get a system of confocal ellipsoids, F being the focal ellipse.]

In a note by Joachimsthal, published since his death, *Crelle* 73, p. 207, he observes, with a similar analogy to the ellipse, that the normal to the ellipsoid is constructed by measuring from d on da, db, dc lengths da', db', dc' which would represent equilibrating forces if measured from D along DA, DB, DC. The resultant of da', db', dc' is the normal of the ellipsoid.

[Let $AD = ad = r_1$, $BD = bd = r_2$, $CD = cd = r_3$, and let $f(r_1, r_2, r_3) = 0$, be the identical relation connecting r_1, r_2, r_3 as D moves in the plane. Then $\frac{df}{dr_1}dr_1 + \frac{df}{dr_2}dr_2 + \frac{df}{dr_3}dr_3 = 0$. Since this holds for all small displacements of D in the plane, it follows by the statical principle of virtual work, that

$$\frac{df}{dr_1}, \frac{df}{dr_2}, \frac{df}{dr_3}$$

are proportional to the unique system of equilibrating forces R_1, R_2, R_3, acting along DA, DB, DC. Now consider the point d; since

$$R_1dr_1 + R_2dr_2 + R_3dr_3 = 0$$

for all small displacements of d along its surface locus, the resultant of the forces R_1, R_2, R_3 along ad, bd, cd must be normal to this locus.]

192. Conversely, let it be required *to find the locus of a point whose distances from three fixed points are connected by the same relation as that which connects the distances from the vertices of a triangle, whose sides are a, b, c, to any point in its plane.* Let ρ, ρ', ρ'' be the three distances, then (Art. 54) the relation which connects them is

$$a^2(\rho^2 - \rho'^2)(\rho^2 - \rho''^2) + b^2(\rho'^2 - \rho^2)(\rho'^2 - \rho''^2) + c^2(\rho''^2 - \rho^2)(\rho''^2 - \rho'^2)$$
$$- a^2(b^2 + c^2 - a^2)\rho^2 - b^2(c^2 + a^2 - b^2)\rho'^2 - c^2(a^2 + b^2 - c^2)\rho''^2$$
$$+ a^2b^2c^2 = 0.$$

But $\rho^2 - \rho'^2$, &c. being only functions of the coordinates of the first degree, the locus is manifestly only of the second degree.

That any of the points from which the distances are measured is a focus, is proved by showing that this equation is of the form $U + LM = 0$, where U is the infinitely small sphere whose centre is this point. Let P and Q represent respectively the linear quantities $\rho'^2 - \rho^2 - c^2$, $\rho''^2 - \rho^2 - b^2$,

and consider the difference between the left-hand side of the equation and the quantity

$$a^2PQ + (b^2P - c^2Q)(P - Q)$$

or

$$b^2P^2 - 2bcPQ\cos A + c^2Q^2,$$

where A is the angle opposite a in the triangle ABC. This difference must be a constant multiple of ρ^2, for it is of the second degree in x, y, z, and it vanishes when ρ^2 vanishes; hence the equation of the locus is of the form stated, and, since the planes L, M are imaginary, the point is a focus of the modular kind.

193. *If several parallel tangent planes touch a series of confocals, the locus of their points of contact is an equilateral hyperbola.*

Let α, β, γ be the direction-angles of the perpendicular on the tangent planes. Then the direction-cosines of the radius vector to any point of contact are $\frac{a^2\cos\alpha}{rp}$, $\frac{b^2\cos\beta}{rp}$, $\frac{c^2\cos\gamma}{rp}$; as easily appears by substituting in the formula $a^2\cos\alpha = px'$ (Art. 89), $r\cos\alpha'$ for x' and solving for $\cos\alpha'$. Forming then, by Art. 15, the direction-cosines of the perpendicular to the plane of the radius vector and the perpendicular on the tangent plane, we find them to be

$$\frac{(b^2-c^2)\cos\beta\cos\gamma}{rp\sin\phi},\quad \frac{(c^2-a^2)\cos\gamma\cos\alpha}{rp\sin\phi},\quad \frac{(a^2-b^2)\cos\alpha\cos\beta}{rp\sin\phi},$$

where ϕ is the angle between the radius vector and the perpendicular. Now the denominator is double the area of the triangle of which the radius vector and perpendicular are sides. Double the projections, therefore, of this triangle on the coordinate planes are

$$(b^2-c^2)\cos\beta\cos\gamma,\ (c^2-a^2)\cos\gamma\cos\alpha,\ (a^2-b^2)\cos\alpha\cos\beta.$$

Now these projections being constant for a system of confocal surfaces, we learn that for such a system, both the plane of the triangle and its magnitude is constant. If then CM be the perpendicular on the series of parallel tangent planes and PM the perpendicular on that line from any point of contact P, we have proved that the plane and the magnitude of the

triangle CPM are constant, and therefore the locus of P is an equilateral hyperbola of which CM is an asymptote.

193a. Writing down the equations of the normals to

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1,$$

at two points, we find as the conditions that they may intersect

$$A\,(x'-x'')\,(y'z''-y''z')+B\,(y'-y'')\,(z'x''-z''x') \\ +C\,(z'-z'')\,(x'y''-x''y')=0,$$

or, calling α, β, γ the direction angles of the line which joins the points, and α_1, β_1, γ_1 those of the perpendicular to the central plane containing the two points, the condition becomes

$$A\cos\alpha\cos\alpha_1+B\cos\beta\cos\beta_1+C\cos\gamma\cos\gamma_1=0.$$

This relation obviously still holds if A, B, C be replaced by $kA+l$, $kB+l$, $kC+l$. Hence we see that *if the normals at the two points of intersection of any right line with any central quadric intersect, the normals at its two points of intersection with any confocal, or with a similar and similarly placed concentric quadric likewise intersect.**

As a special case of this, we may consider the *three confocals*, u, v, w, which meet in any point P. The normal at P to u meets u again in Q, therefore meets the normal at Q. Hence, if normals be drawn to v at the points in which it is met by PQ they must intersect, and, in like manner, the normals at the points where PQ meets w, intersect. But the line PQ is a tangent line both to v and to w. Hence, *normals to either surface taken at consecutive points along their common curve intersect.* A curve possessing this property is defined to be a *line of curvature* on either surface.

Curvature of Quadrics.

194. The general theory of the curvature of surfaces will be explained in Chap. XI., but it will be convenient to state

* See a paper by F. Purser, *Quarterly Journal of Mathematics*, p. 66, Vol. VIII.

here some theorems on the curvature of quadrics which are immediately connected with the subject of this chapter.

If a normal section be made at any point on a quadric, its radius of curvature at that point is equal to $\beta^2 : p$, where β is the semi-diameter parallel to the trace of the section on the tangent plane, and p is the perpendicular from the centre on the tangent plane.

We repeat the following proof by the method of infinitesimals from *Conics*, Art. 398, which see.

Let P, Q be any two points on a quadric; let a plane through Q parallel to the tangent plane at P meet the central radius CP in R, and the normal at P in S, then the radius of a circle through the points P, Q having its centre on PS is $PQ^2 : 2PS$. But if the point Q approach indefinitely near to P, QP is in the limit equal to QR; and if we denote CP and the central radius parallel to QR by a' and β, and if P' be the other extremity of the diameter CP, then (Art. 74)

$$\beta^2 : a'^2 :: QR^2 : PR \,.\, RP' \; (= 2a' \,.\, PR);$$

therefore $QR^2 = \dfrac{2\beta^2 \,.\, PR}{a'}$ and the radius of curvature $= \dfrac{\beta^2}{a'} \cdot \dfrac{PR}{PS}$.
But if from the centre we let fall a perpendicular CM on the tangent plane, the right-angled triangle CMP is similar to PRS, and $PR : PS :: a' : p$. And the radius of curvature is therefore $\dfrac{\beta^2}{a'} \cdot \dfrac{a'}{p} = \dfrac{\beta^2}{p}$; which was to be proved.

If the circle through PQ have its centre not on PS, but on any line PS', making an angle θ with PS, the only change is that the radius of the circle is $\dfrac{PQ^2}{2PS'}$, S' being still on the plane drawn through Q parallel to the tangent plane at P. But PS evidently $= PS' \cos\theta$. The radius of curvature is therefore $\dfrac{PQ^2}{2PS}\cos\theta$, or *the value for the radius of curvature of an oblique section is the radius of curvature of the normal section through PQ, multiplied by* $\cos\theta$. (Meunier's theorem.)

195. These theorems may also easily be proved analytically. It is proved (*Conics*, Art. 241) that if $ax^2 + 2hxy + by^2 + 2gx = 0$

be the equation of any conic, the radius of curvature at the origin is $g \div b$. If then the equation of any quadric, the plane of xy being a tangent plane, be

$$ax^2 + 2hxy + by^2 + 2gzx + 2fyz + cz^2 + 2nz = 0,$$

the radii of curvature by the sections $y = 0$, $x = 0$ are respectively $n : a$, $n : b$. But if the equation be transformed to parallel axes through the centre, the terms of highest degree remain unaltered, and the equation becomes

$$ax^2 + 2hxy + by^2 + 2gzx + 2fyz + cz^2 = D.$$

The squares of the intercepts on the axes of x and y are $D : a$, $D : b$. This proves that the radii of curvature are proportional to the squares of the parallel semi-diameters of a central section. And since, by the theory of conics, the radius of curvature of that section which contains the perpendicular on the tangent plane is $\beta^2 : p$, the same is the form of the radius of every other section.

The same may be proved by using the equation of the quadric transformed to any normal and the normals to two confocals as axes (see Ex. 2, Art. 174), viz.

$$\frac{x^2}{\gamma^2} + \frac{y^2}{a^2 - a'^2} + \frac{z^2}{a^2 - a''^2} - \frac{2p'xy}{p(a^2 - a'^2)} - \frac{2p''xz}{p(a^2 - a''^2)} + \frac{2x}{p} = 0.$$

The radii of curvature of the sections by the planes $z = 0$, $y = 0$ are respectively $\dfrac{a^2 - a'^2}{p}$, $\dfrac{a^2 - a''^2}{p}$. The numerators are the squares of the semi-axes of the section by a plane parallel to the tangent plane (Art. 164).

The equation of the section made by a plane making an angle θ with the plane of y is found by first turning the axes of coordinates round through an angle θ, by substituting $y \cos\theta - z \sin\theta$, $y \sin\theta + z\cos\theta$ for y and z, and then making the new $z = 0$. Then, if the new coefficient of y^2 is $\dfrac{1}{\beta^2}$, $\dfrac{\beta^2}{p}$ is the corresponding radius of curvature. But this coefficient is at once found to be

$$\frac{\cos^2\theta}{a^2 - a'^2} + \frac{\sin^2\theta}{a^2 - a''^2},$$

and this coefficient of y^2 is evidently the inverse square of

that semi-diameter of the central section, which makes an angle θ with the axis y.

196. It follows from the theorem enunciated in Art. 194, that *at any point on a central quadric the radius of curvature of a normal section has a maximum and minimum value, the directions of the sections for these values being parallel to the axis-major and axis-minor of the central section by a plane parallel to the tangent plane.*

These maximum and minimum values are called the *principal radii of curvature* for that point, and the sections to which they belong are called the *principal sections.* It appears (from Art. 163) that the principal sections contain each the normal to one of the confocals through the point. *The intersection of a quadric with a confocal is a curve such that at every point of it the tangent to the curve is one of the principal directions of curvature.* Such a curve is called a *line of curvature* on the surface, and this definition agrees with that of Art. 193*a*.

In the case of the hyperboloid of one sheet the central section is a hyperbola, and the sections whose traces on the tangent plane are parallel to the asymptotes of that hyperbola will have their radii of curvature infinite; that is to say, they will be right lines, as we know already. In passing through one of those sections the radius of curvature changes sign; that is to say, the direction of the convexity of sections on one side of one of those lines is opposite to that of those on the other.

197. *The two principal centres of curvature are the two poles of the tangent plane with regard to the two confocal surfaces which pass through the point of contact.* For these poles lie on the normal to that plane (Art. 167), and at distances from it $=\dfrac{a^2-a'^2}{p}$, and $\dfrac{a^2-a''^2}{p}$ (Art. 168), but these have been just proved to be the lengths of the principal radii of curvature.

We can also hence find, by Art. 168, the coordinates of the centres of the two principal circles of curvature, viz.

$$x = \frac{a'^2 x'}{a^2},\ y = \frac{b'^2 y'}{b^2},\ z = \frac{c'^2 z'}{c^2};\quad x = \frac{a''^2 x'}{a^2},\ y = \frac{b''^2 y'}{b^2},\ z = \frac{c''^2 z'}{c^2}.$$

198. If at each point of a quadric we take the two principal centres of curvature, the locus of all these centres is a surface of two sheets, which is called the *surface of centres.*

We shall show how to find its equation in the next chapter, but we can see *à priori* the nature of its *sections by the principal planes.* In fact, one of the principal radii of curvature at any point on a principal section is the radius of curvature of the section itself, and the locus of the centres corresponding is evidently the evolute of that section. The other radius of curvature corresponding to any point in the section by the plane of xy is $c^2 : p$, as appears from the formula of Art. 194, since c is an axis in every section drawn through the axis of z. From the formulæ of Art. 197 the coordinates of the corresponding centre are $\frac{a^2 - c^2}{a^2} x'$, $\frac{b^2 - c^2}{b^2} y'$; that is to say, they are the poles with regard to the focal conic of the tangent at the point $x'y'$ to the principal section. The locus of the centres will be the reciprocal of the principal section, taken with regard to the focal conic, viz.

$$\frac{a^2 x^2}{(a^2 - c^2)^2} + \frac{b^2 y^2}{(b^2 - c^2)^2} = 1.$$

The section then by a principal plane of the surface (which is of the twelfth degree) consists of the evolute of a conic, which is of the sixth degree, and of a conic (it will be found) three times over, this conic being a cuspidal line on the surface. The section by the plane at infinity is of a similar nature to those of the principal planes. It may be added, that the conic touches the evolute in four points (real for the principal plane through the greatest and least axes, or umbilicar plane) and besides cuts it in four points.

199. *The spherical reciprocal, with regard to the centre, of the surface of centres is a surface of the fourth degree.*

It will appear from the general theory of the curvature of surfaces, to be explained in Chap. XI., that the tangent plane to either of the confocal surfaces through $x'y'z'$ is also a tangent plane to the surface of centres. The reciprocals of the intercepts which the tangent plane makes on the axes are given by the equations

$$\xi = \frac{x'}{a'^2},\ \eta = \frac{y'}{b'^2},\ \zeta = \frac{z'}{c'^2}.$$

The relation

$$\frac{x'^2}{a^2a'^2} + \frac{y'^2}{b^2b'^2} + \frac{z'^2}{c^2c'^2} = 0$$

gives between ξ, η, ζ the relation

$$\xi^2 + \eta^2 + \zeta^2 = (a^2 - a'^2)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right),$$

and the relation

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} = 1$$

gives $\quad a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1 = (a^2 - a'^2)(\xi^2 + \eta^2 + \zeta^2).$

Eliminating $a^2 - a'^2$, we have

$$(\xi^2 + \eta^2 + \zeta^2)^2 = \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right)(a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1).*$$

But it is evident (as at *Higher Plane Curves*, Art. 21) that ξ, η, ζ may be understood to be coordinates of the reciprocal surface; since, if ξ, η, ζ be the coordinates of the pole of the tangent plane with regard to the sphere $x^2 + y^2 + z^2 = 1$, the equation $x\xi + y\eta + z\zeta = 1$ being identical with that of the tangent plane, ξ, η, ζ will be also the reciprocals of the intercepts made by the tangent plane on the axis.

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* This equation was first given, as far as I am aware, by Booth, *Tangential Coordinates*, Dublin, 1840.

CHAPTER IX.

INVARIANTS AND COVARIANTS OF SYSTEMS OF QUADRICS.

Systems of Two Quadrics.

200. It was proved (Art. 136) that there are four values of λ for which $\lambda U + V$ represents a cone. The biquadratic which determines λ is obtained by equating to nothing the discriminant of $\lambda U + V$. We shall write it

$$\lambda^4\Delta + \lambda^3\Theta + \lambda^2\Phi + \lambda\Theta' + \Delta' = 0.$$

The values of λ, for which $\lambda U + V$ represents a cone, are evidently independent of the system of coordinates in which U and V are expressed. The coefficients Δ, Θ, &c. are therefore *invariants* whose mutual ratios are unaltered by transformation of coordinates.* The following exercises in calculating these invariants include some of the cases of most frequent occurrence.

Ex. 1. Let both quadrics be referred to their common self-conjugate tetrahedron. We may take

$$U = ax^2 + by^2 + cz^2 + dw^2,\ V = x^2 + y^2 + z^2 + w^2,$$

(see Art. 141, and *Conics*, Ex. 1, Art. 371), then

$$\Delta = abcd,\ \Theta = bcd + cda + dab + abc,\ \Phi = bc + ca + ab + ad + bd + cd,$$
$$\Theta' = a + b + c + d,\ \Delta' = 1.$$

Ex. 2. Let V, as before, be $x^2 + y^2 + z^2 + w^2$, and let U represent the general equation. The general value of Θ is

[* If U be a homogeneous quantic of the second order in n variables $(x_1, x_2, x_3 \ldots x_n)$, say $U \equiv a_{11}x_1^2 + a_{22}x_2^2 + \ldots + 2a_{12}x_1x_2 + \ldots$ then the determinant Δ, whose vanishing is the condition that the n quantities $\frac{dU}{dx_1}, \frac{dU}{dx_2}$, &c., may simultaneously vanish for some value of the coordinates is an invariant of U. $\Delta = 0$ in general is the condition that U may be expressed in $n - 1$ homogeneous coordinates. And as in the text we construct other invariants of the system $\lambda U + V$, where V is another homogeneous quantic of the second order.]

$$a'A + b'B + c'C + d'D + 2f'F + 2g'G + 2h'H + 2l'L + 2m'M + 2n'N,$$

where A, B, &c. have the same meaning as in Art. 67. In the present case therefore

$$\Theta = A + B + C + D, \Theta' = a + b + c + d;$$

we have also $\Phi = bc - f^2 + ca - g^2 + ab - h^2 + ad - l^2 + bd - m^2 + cd - n^2$.

Similarly, if U is $ax^2 + by^2 + cz^2 + dw^2$, and V is the general equation,

Θ is $a'bcd + b'cda + c'dab + d'abc$, Θ' is $aA' + bB' + cC' + dD'$.

Ex. 3. Let U and V represent two spheres,

$$x^2 + y^2 + z^2 - \rho^2, (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - \rho'^2,$$

and let the distance between the centres be D, $(\alpha^2 + \beta^2 + \gamma^2 = D^2)$, then

$\Delta = -\rho^2$, $\Delta' = -\rho'^2$, $\Theta = D^2 - 3\rho^2 - \rho'^2$, $\Theta' = D^2 - \rho^2 - 3\rho'^2$, $\Phi = 2D^2 - 3\rho^2 - 3\rho'^2$,

and the biquadratic which determines λ is

$$(\lambda + 1)^2 \{-\rho^2\lambda^2 + (D^2 - \rho^2 - \rho'^2)\lambda - \rho'^2\} = 0.$$

Ex. 4. Let U represent $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, while V is the sphere

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - \rho^2.$$

$$Ans.\ \Delta = -\frac{1}{a^2b^2c^2}, \Delta' = -\rho^2,$$

$$\Theta = \frac{1}{a^2b^2c^2}\{\alpha^2 + \beta^2 + \gamma^2 - \rho^2 - (a^2 + b^2 + c^2)\}.$$

$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 - \rho^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right),$$

$$\Phi = \frac{1}{b^2c^2}(\beta^2 + \gamma^2 - \rho^2) + \frac{1}{c^2a^2}(\gamma^2 + \alpha^2 - \rho^2)$$
$$+ \frac{1}{a^2b^2}(\alpha^2 + \beta^2 - \rho^2) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Since $\lambda U + V$ admits of being written in the form

$$AX^2 + BY^2 + CZ^2 + DW^2,$$

it is easily seen that the biquadratic found by equating to nothing the discriminant of $\lambda U + V$ may be written

$$\frac{\alpha^2}{a^2 + \lambda} + \frac{\beta^2}{b^2 + \lambda} + \frac{\gamma^2}{c^2 + \lambda} = 1 + \frac{\rho^2}{\lambda}.$$

Ex. 5. Let U represent the paraboloid $ax^2 + by^2 + 2nz$ and V the sphere as in the last example.

$$Ans.\ \Delta = -abn^2, \Delta' = -\rho^2,$$

$$\Theta = -n^2(a + b) + 2abn\gamma, \Theta' = a\alpha^2 + b\beta^2 + 2n\gamma - (a + b)\rho^2,$$

$$\Phi = ab(\alpha^2 + \beta^2 - \rho^2) + 2(a + b)n\gamma - n^2;$$

and the biquadratic may be written by a similar method

$$\frac{\lambda a\alpha^2}{\lambda a + 1} + \frac{\lambda b\beta^2}{\lambda b + 1} + 2\lambda n\gamma = \lambda^2n^2 + \rho^2.$$

Ex. 6. In general the value of Φ is

$$(bc - f^2)(a'd' - l'^2) + (ca - g^2)(b'd' - m'^2) + (ab - h^2)(c'd' - n'^2)$$
$$+ (ad - l^2)(b'c' - f'^2) + (bd - m^2)(c'a' - g'^2) + (cd - n^2)(a'b' - h'^2)$$
$$+ 2(gm - hn)(g'm' - h'n') + 2(hn - fl)(h'n' - f'l') + 2(fl - gm)(f'l' - g'm')$$
$$+ 2(mh - lb)(l'c' - n'g') + 2(nf - mc)(m'a' - l'h') + 2(lg - na)(n'b' - m'f')$$
$$+ 2(m'h' - l'b')(lc - ng) + 2(n'f' - m'c')(ma - lh) + 2(l'g' - n'a')(nb - mf)$$
$$+ 2(fd - mn)(g'h' - a'f') + 2(gd - nl)(h'f' - b'g') + 2(hd - lm)(f'g' - c'h')$$
$$+ 2(f'd' - m'n')(gh - af) + 2(g'd' - n'l')(hf - bg) + 2(h'd' - l'm')(fg - ch).$$

Thus Φ is a function of the same quantities which occur in the condition (Art. 80c) that a line should touch a quadric. This condition is a quadratic function of the six coordinates of the line; and if we write the coefficients which affect the squares of the coordinates in that condition $a_{11}, a_{22} \ldots a_{66}$, and those which affect the double rectangles a_{12}, a_{13}, &c., writing the corresponding quantities for the second quadric c_{11}, c_{22}, &c., then Φ is

$$a_{11}c_{44} + a_{22}c_{55} + a_{33}c_{66} + a_{44}c_{11} + a_{55}c_{22} + a_{66}c_{33} + 2a_{14}c_{11} + \&c.$$

In like manner, writing the discriminant in any of the three forms,

$$\begin{aligned}\Delta &= a_{11}a_{44} + a_{12}a_{45} + a_{13}a_{46} + a^2_{14} + a_{15}a_{42} + a_{16}a_{43} \\ &= a_{21}a_{54} + a_{22}a_{55} + a_{23}a_{56} + a_{24}a_{51} + a^2_{25} + a_{26}a_{53} \\ &= a_{31}a_{64} + a_{32}a_{65} + a_{33}a_{66} + a_{34}a_{61} + a_{35}a_{62} + a^2_{36},\end{aligned}$$

if by the substitution of $a + \lambda a'$ &c. for a &c., a_{11} become $a_{11} + \lambda b_{11} + \lambda^2 c_{11}$ &c., different methods of writing the invariants are found.

201. *To examine the geometrical meaning of the condition* $\Theta = 0$ *and of the condition* $\Phi = 0$. It appears, from Art. 200, Ex. 2, that when U is referred to a self-conjugate tetrahedron,

$$\Theta = bcda' + cdab' + dabc' + abcd',$$

which will vanish when a', b', c', d' all vanish. Hence Θ *will vanish whenever it is possible to inscribe in* V *a tetrahedron which shall be self-conjugate with regard to* U. In like manner Θ' will vanish for this form of U whenever A', B', C', D' vanish. But $A' = 0$ is the condition that the plane x shall touch V. Hence Θ' *will vanish whenever it is possible to find a tetrahedron self-conjugate with regard to* U *whose faces touch* V. By the first part of this article $\Theta' = 0$ is also the condition that it may be possible to inscribe in U a tetrahedron self-conjugate with regard to V. Hence when one of these things is possible, so is the other also.

$\Phi = 0$ will be fulfilled, if the edges of a self-conjugate tetrahedron, with respect to either, all touch the other.

Ex. 1. The vertices of two self-conjugate tetrahedra, with respect to a quadric form a system of eight points, such that every quadric through seven will pass through the eighth (Hesse, *Crelle*, Vol. XX., p. 297).

Let any quadric be described through the four vertices of one tetrahedron, and through three vertices of the second, whose faces we take for x, y, z, w. Then because the quadric circumscribes the first tetrahedron, $\Theta' = 0$, or $a + b + c + d = 0$ (Art. 200, Ex. 2); and because it passes through three vertices of $xyzw$, we have $a = 0, b = 0, c = 0$; therefore $d = 0$, or the quadric goes through the remaining vertex. It is proved, in like manner, that any

quadric which touches seven of the faces of the two tetrahedra touches the eighth.

Ex. 2. If a sphere be circumscribed about a self-conjugate tetrahedron, the length of the tangent to it from the centre of the quadric is constant. For (Art. 200, Ex. 4) the condition $\Theta = 0$ gives the square of the tangent

$$\alpha^2 + \beta^2 + \gamma - \rho^2 = a^2 + b^2 + c^2.$$

This corresponds to Faure's theorem (*Conics*, Art. 375, Ex. 2). It may be otherwise stated: "The sphere which circumscribes a self-conjugate tetrahedron cuts orthogonally the sphere which is the intersection of three tangent planes at right angles" (Art. 93).

Ex. 3. If a hyperboloid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$ be such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$, then the centre of a sphere inscribed in a self-conjugate tetrahedron lies on the surface. This follows from the condition $\Theta' = 0$ (Art. 200, Ex. 4).

Ex. 4. The locus of the centre of a sphere circumscribing a tetrahedron, self-conjugate with regard to a paraboloid, is a plane (Art. 200, Ex. 5).

202. *To find the condition that two quadrics of the system $\lambda U + V$ should touch each other.* As in the case of conics (*Conics*, Art. 372) the biquadratic of Art. 200 will have *two equal roots* when the quadrics U, V touch. This is most easily proved by taking the origin at the point of contact, and the tangent plane for the coordinate plane z. Then, for both the quadrics, we have $d = 0$, $l = 0$, $m = 0$; and since, if we substitute these values in the discriminant (Art. 67), it reduces to $n^2 (h^2 - ab)$, the biquadratic becomes

$$(\lambda n + n')^2 \{(\lambda h + h')^2 - (\lambda a + a')(\lambda b + b')\} = 0,$$

which has two equal roots. The required condition is therefore found by equating to zero the discriminant of the biquadratic of Art. 200. [When this condition is fulfilled, *any* two quadrics of the system $\lambda U + V$ touch each other, for the result of substituting any two such quadrics for U and V is equivalent to substituting $\frac{p\lambda + q}{r\lambda + s}$ for λ.]

Generally the biquadratic (Art. 200) will have *two pairs of equal roots when the quadrics have a generator common;* the conditions for this may be written down as in *Higher Algebra*, Art. 138, or Burnside and Panton, Art. 68.*

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* For a complete classification of the simplest forms of specially related quadrics of system $\lambda U + V = 0$, see Bromwich, *Quadratic Forms*. p. 46.

Ex. 1. To find the condition that two spheres may touch. The biquadratic for this case (Art. 200, Ex. 3) has always two equal roots. This is because two spheres having common a plane section at infinity, always have double contact at infinity (Art. 137). The condition that they should besides have finite contact is got by expressing the condition that the other two factors of the biquadratic should be equal and is $(D^2 - r^2 - r'^2)^2 = 4r^2r'^2$, whence $D = r \pm r'$.

Ex. 2. Find the locus of the centre of a sphere of constant radius touching a central quadric. The equation got by forming the discriminant with respect to λ of the biquadratic of Art. 200, Ex. 4, is of the twelfth degree in α, β, γ. When we make $\rho = 0$, it reduces to the quadric taken twice, together with the equation of the eighth degree considered below (Art. 221).

The problem considered in this example is the same as that of finding the equation of the surface *parallel* to the quadric (see *Conics*, Ex. 3, Art. 372); namely, the surface generated by measuring from the surface on each normal a constant length equal to ρ. The equation is of the sixth degree in ρ^2, and gives the lengths of the *six* normals which can be drawn from any point xyz to the surface (*Conics*, Art. 372, Ex. 3). To find the section of the surface by one of the principal planes, we use the principle that the discriminant with respect to λ of any algebraic expression of the form $(\lambda - a)\,\phi(\lambda)$ is the square of $\phi(a)$ multiplied by the discriminant of $\phi(\lambda)$. If then we make $z = 0$ in the equation, the discriminant of

$$(\lambda + c)\left\{\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1 - \frac{\rho^2}{\lambda}\right\}$$

is the conic
$$\frac{x^2}{a-c} + \frac{y^2}{b-c} - 1 + \frac{\rho^2}{c},$$

taken twice, this curve being a double line on the surface, together with the discriminant of the function within the brackets; this latter representing the curve of the eighth order, parallel to the principal section of the ellipsoid.

[Ex. 3. To express the coordinates of the parallel surface by means of two parameters.

The discriminant is to have two equal roots; let these be p, and let q, r be the other roots. Then

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - \frac{R^2}{\lambda} - 1 \equiv \frac{-(\lambda - p)^2(\lambda - q)(\lambda - r)}{\lambda(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)}.$$

Multiplying across by λ and putting $\lambda = 0$ we get $R^2 = \dfrac{p^2qr}{a^2b^2c^2}$, which expresses one parameter in terms of the other two. Multiplying by $\lambda + a^2$ and putting $\lambda = -a^2$, gives
$$x^2 = \frac{(p+a^2)^2(q+a^2)(r+a^2)}{a^2(b^2-a^2)(c^2-a^2)}\ \&c.$$

Using this we can find the sections by the principal planes.

If a', a'' are semi-axes of the confocals passing through the point x', y', z' at which the sphere whose centre is x, y, z touches the quadric whose semi-axis is a, then $q = a'^2 - a^2, r = a''^2 - a^2$. This may be proved by finding the coordinates of a point (P) on the normal at $x'y'z'$ (P') such that $PP' = R$, and using Arts. 160 and 165.]

Ex. 4. The equation of the surface *parallel to a paraboloid* is found in like

manner by forming the discriminant of the biquadratic of Ex. 5, Art. 200. The result represents a surface of the tenth degree, and when $\rho = 0$, reduces to the paraboloid taken twice, together with the surface of the sixth degree considered below (Art. 222). The equation is of the fifth degree in ρ^2, showing that only five normals can be drawn from any point to the surface. It is seen, as in the last example, that the section by either principal plane is a parabola taken twice, together with the curve parallel to a parabola.

203. It is to be remarked that *when two surfaces touch the point of contact is a double point on their curve of intersection.* In general, two surfaces of the m^{th} and n^{th} degrees respectively intersect in a curve of the mn^{th} order. And at each point of the curve of intersection there is a single tangent line, namely the intersection of the tangent planes at that point to the two surfaces. For any plane drawn through this line meets the surfaces in two curves which touch : such a plane therefore passes through two coincident points of the curve of intersection. But if the surfaces touch, then *every* plane through the point of contact meets them in two curves which touch, and *every* such plane therefore passes through two coincident points of the curve of intersection. The point of contact is therefore a double point on this curve.

And we can show that, as in plane curves, there are two tangents at the double point. For there are two directions in the common tangent plane to the surfaces, any plane through either of which meets the surfaces in curves having three points in common.

Take the tangent plane for the plane of xy, and let the equations of the surfaces be

$$z + ax^2 + 2hxy + by^2 + \&c.,$$
$$z + a'x^2 + 2h'xy + b'y^2 + \&c.,$$

then any plane $y = \mu x$ cuts the surfaces in curves which osculate (see *Conics*, Art. 239), if

$$a + 2h\mu + b\mu^2 = a' + 2h'\mu + b'\mu^2.$$

The two required directions then are given by the equation

$$(a - a')x^2 + 2(h - h')\,xy + (b - b')y^2 = 0.$$

The same may be otherwise proved thus. It will be shown hereafter precisely as at *Higher Plane Curves*, Arts.

36, 37, that if the equation of a surface be $u_1 + u_2 + u_3 + \&c. = 0$, the origin will be on the surface, and u_1 will include all the right lines which meet the surface in two consecutive points at the origin; while if u_1 is identically 0, the surface has the origin for a double point, and u_2 includes all the right lines which meet the surface at the origin in three consecutive points. Now in the case we are considering, by subtracting one equation from the other, we get a surface through the curve of intersection, viz.

$$(a-a')x^2 + 2(h-h')xy + (b-b')y^2 + \&c.,$$

in which surface the origin is a double point, and the two lines just written are two lines which meet the surface in three consecutive points.

204. When these lines coincide there is a cusp or stationary point (see *Higher Plane Curves*, Art. 38) on the curve of intersection. We shall call the contact in this case *stationary* contact. The condition that this should be the case, the axes being assumed as above, is

$$(a-a')\ (b-b') = (h-h')^2.$$

Now if we compare the biquadratic for λ, given Art. 202, remembering also that in the form we are now working with, we have $n = n'$, we shall see that when this condition is fulfilled, three roots of the biquadratic become equal to -1. *The conditions then for stationary contact are found by forming the conditions that the biquadratic should have three equal roots*, viz. these conditions are $S = 0$, $T = 0$, S and T being the two invariants of the biquadratic.

[When these conditions are fulfilled *any* two quadrics of the system have stationary contact.]

[Ex. 1. In general four quadrics of the system reduce to cones. What becomes of these cones when (a) the quadrics touch, (b) have a common generator, and (c) have stationary contact? How are two cones of the system related in each case?

Ex. 2. Reciprocate the results of Ex. 1.]

205. Every sphere whose centre is on a normal to a quadric, and which passes through the point where the

normal meets the surface, of course touches the surface. But it will have *stationary* contact when the length of the radius of the sphere is equal to one of the *principal* radii of curvature (Art. 196). Let us take the tangent plane for plane of xy, and the two directions of maximum and minimum curvature (Art. 196) for the axes of x and y. Then since these directions are parallel to the axes of parallel sections, the term xy will not appear in the equation, which will be of the form $z + ax^2 + by^2 + \&c. = 0$. By the last article, any sphere $z + \lambda\ (x^2 + y^2 + z^2)$ will have stationary contact with this if $(\lambda - a)(\lambda - b) = 0$, for we have h and h' each $= 0$. We must therefore have λ equal either to a or b. Now if we make $y = 0$, the circle $z + a\ (x^2 + z^2)$ is evidently that which osculates the section $z + ax^2 + \&c.$; and, in like manner, the circle $z + b\ (y^2 + z^2)$ osculates $z + by^2 + \&c.$

206. *To find the equation of the surface of centres of a quadric.* If we form, for the biquadratic of Ex. 4, Art. 200, the two equations $S = 0$, $T = 0$, we have two equations connecting α, β, γ, the coordinates of the centre of curvature of any principal section, and ρ its radius. One of these equations is a quadratic and the other a cubic in ρ^2; and if we eliminate ρ^2 between them, we evidently have the equation of the locus of the centres of curvature of all principal sections. The problem may also be stated thus: If U and U' be any two algebraical expressions of the same degree and k a variable parameter, it is generally possible to determine k so that the equation $U + kU' = 0$ may have two equal roots. But it is not possible to determine k, so that the same equation may have three equal roots, unless a certain invariant relation subsist between the coefficients of U and U'. Now the present problem is a particular case of the general problem of finding such an invariant relation. It is in fact to find the condition that it may be possible to determine k so that the following biquadratic in λ may have three equal roots:

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 + \frac{k}{\lambda}.$$

The following are the leading terms in the resulting equation: the remaining terms can be added from the symmetry of the letters. We use the abbreviations $b^2 - c^2 = \alpha$, $c^2 - a^2 = \beta$, $a^2 - b^2 = \gamma$; and further we write x, y, z instead of ax, by, cz:

$$\begin{aligned}
&\alpha^6x^{12} + 3(\alpha^2+\beta^2)\alpha^4x^{10}y^2 + 3(\alpha^4+3\alpha^2\beta^2+\beta^4)\alpha^2x^8y^4 \\
&+ 3(2\alpha^4+3\alpha^2\beta^2+3\alpha^2\gamma^2-7\beta^2\gamma^2)\alpha^2x^8y^2z^2 \\
&+ (\alpha^6+\beta^6+9\alpha^4\beta^2+9\alpha^2\beta^4)x^6y^6 \\
&+ 3(\alpha^6+6\alpha^4\beta^2+3\alpha^4\gamma^2+3\alpha^2\beta^4+\beta^4\gamma^2-21\alpha^2\beta^2\gamma^2)x^6y^4z^2 \\
&+ 9(\alpha^4\beta^2+\beta^4\alpha^2+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4-14\alpha^2\beta^2\gamma^2)x^4y^4z^4 \\
&- 3(\beta^2+\gamma^2)\alpha^6x^{10} - 3(2\beta^4+3\beta^2\gamma^2+3\beta^2\alpha^2-7\gamma^2\alpha^2)\alpha^4x^8y^2 \\
&- 3(\beta^6+6\beta^4\alpha^2+3\beta^4\gamma^2+3\beta^2\alpha^4+\alpha^4\gamma^2-21\alpha^2\beta^2\gamma^2)\alpha^2x^6y^4 \\
&+ 3\{14(\alpha^4\beta^2+\alpha^2\beta^4+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4)+20\alpha^2\beta^2\gamma^2\}\alpha^2x^6y^2z^2 \\
&+ 3\{4\gamma^8-7\gamma^6(\alpha^2+\beta^2)-198\gamma^4\alpha^2\beta^2+68\alpha^2\beta^2\gamma^2(\alpha^2+\beta^2)+42\alpha^4\beta^4\}x^4y^4z^2 \\
&+ 3(\beta^4+3\beta^2\gamma^2+\gamma^4)\alpha^6x^8 \\
&+ 3(\beta^6+6\beta^4\gamma^2+3\beta^4\alpha^2+3\beta^2\gamma^4+\alpha^2\gamma^4-21\alpha^2\beta^2\gamma^2)\alpha^4x^6y^2 \\
&+ 9(\alpha^4\beta^2+\alpha^2\beta^4+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4-14\alpha^2\beta^2\gamma^2)\alpha^2\beta^2x^4y^4 \\
&- 3\{4\alpha^8-7\alpha^6(\beta^2+\gamma^2)-198\alpha^4\beta^2\gamma^2+68\alpha^2\beta^2\gamma^2(\beta^2+\gamma^2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 42\beta^4\gamma^4\}\alpha^2x^4y^2z^2 \\
&- (\beta^6+\gamma^6+9\beta^4\gamma^2+9\beta^2\gamma^4)\alpha^6x^6 \\
&- 3(\gamma^6+6\gamma^4\beta^2+3\gamma^4\alpha^2+3\gamma^2\beta^4+\alpha^2\beta^4-21\alpha^2\beta^2\gamma^2)\alpha^4\beta^2x^4y^2 \\
&+ 3\{14(\alpha^4\beta^2+\alpha^2\beta^4+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4)+20\alpha^2\beta^2\gamma^2\}\alpha^2\beta^2\gamma^2x^2y^2z^2 \\
&+ 3(\beta^4+3\beta^2\gamma^2+\gamma^4)\alpha^6\beta^2\gamma^2x^4 \\
&+ 3(2\gamma^4+3\gamma^2\alpha^2+3\gamma^2\beta^2-7\alpha^2\beta^2)\alpha^4\beta^4\gamma^2x^2y^2 \\
&- 3(\beta^2+\gamma^2)\alpha^6\beta^4\gamma^4x^2 + \alpha^6\beta^6\gamma^6 = 0.
\end{aligned}$$

If we make in this equation $z = 0$, we obtain

$$(\alpha^2x^2+\beta^2y^2-\alpha^2\beta^2)^3\{(x^2+y^2-\gamma^2)^3+27x^2y^2\gamma^2\},$$ see Art. 198.

The section by the plane at infinity is of a similar kind to that by the principal planes, the highest terms in the equation being

$$(x^2+y^2+z^2)^3\{(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)^3 - 27\alpha^2\beta^2\gamma^2x^2y^2z^2\}.$$

In like manner we find the *surface of centres of the paraboloid* $ax^2 + by^2 + 2nz$. If we write

$$a - b = m,\ a + b = p,\ ab = q,\ bx^2 + ay^2 = V,\ x^2 + y^2 = \rho^2,$$
$$qz^2 + pnz + n^2 = W,$$

the equation is

$$8\{q^2zV + qn(b^2x^2 + a^2y^2) + 2m^2nW\}^3 + 27T = 0,$$

where

$$T = q^5 n V^4 - 16m^2q^4nWx^2y^2 + 6m^2q^4n^2zV^3 - 56m^2q^5n^2zVx^2y^2$$
$$+ 8m^4q^3n^3x^2y^2W + 12m^4q^3n^3z^2V^2 + 6m^2q^4n^3\rho^2V^2 - 152m^2q^5n^3x^2y^2\rho^2$$
$$+ 48m^2pq^4n^3x^2y^2V + 8m^6q^2n^4z^3V + 24m^4q^3n^4z\rho^2V + 24m^6q^2n^5\rho^2z^2$$
$$+ 12m^4q^3n^5\rho^4 + 48m^6q^2n^5x^2y^2 + 24m^6zn^6q(ax^2 + by^2)$$
$$+ 8m^6(a^2x^2 + b^2y^2)n^7.$$

The section by either plane x or y, is a parabola, counted three times, and the evolute of a parabola.

[The coordinates of a point on the surface of centres may be expressed in terms of two parameters as follows: the biquadratic has three equal roots, therefore we may put

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 - \frac{k}{\lambda} \equiv \frac{(\lambda - p)^3(q - \lambda)}{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

Multiply by $\lambda + a^2$ and put $\lambda = -a^2$, and we get

$$x^2 = \frac{(a^2+p)^3(a^2+q)}{a^2(a^2-b^2)(a^2-c^2)} \text{ \&c.}$$

Hence the sections by the principal planes can be found. If $x = 0, p = -a^2$, counted three times; and $q = -a^2$ counted once, and the sections are expressed in a single parameter. (Cf. Art. 198).

It may be proved (Arts. 197, 160) that $p = a'^2 - a^2$, $q = a''^2 - a^2$ for one sheet, and if a' and a'' are interchanged we get the other sheet, a' and a'' being the semi-axes of confocals through the point on the quadric corresponding to the point on the surface of centres.]

207. *To find the condition that two quadrics shall be such that a tetrahedron can be inscribed in one having two pairs of opposite edges on the surface of the other.** The one quadric then can have its equation thrown into the form $Fyz + Lxw = 0$, while the coefficients a, b, c, d are wanting in the equation of the other. We have, then,

$$\Delta = F^2L^2,\ \Theta = 2FL\,(Fl + Lf),\ \Phi = (Fl + Lf)^2 + 2FL(fl - gm - hn),$$
$$\Theta' = 2(fl - gm - hn)\,(Fl + Lf).$$

And the required condition is

$$4\Delta\Theta\Phi = \Theta^3 + 8\Delta^2\Theta'.$$

* This problem and its reciprocal appear to answer to the plane problem of finding the condition that a triangle can be inscribed in one conic and circumscribed about another. Purser (*Quarterly Journal*, Vol. VIII. p. 149) has determined the envelope of the fourth face of a tetrahedron whose other three faces touch a quadric U when two pairs of its opposite edges are generators of another quadric V to be a quadric passing through the curve of intersection of the given quadrics.

Reciprocally the *condition that it may be possible to find a tetrahedron having two pairs of opposite edges on the surface of one, and whose four faces touch the other*, is

$$4\Delta'\Theta'\Phi = \Theta'^3 + 8\Delta'^2\Theta.$$

This may be derived from the equation examined in the next article.

208. *To find the general form of the equation of a quadric which touches the four faces x, y, z, w of the tetrahedron of reference.* The reciprocal quadric will pass through the four vertices of the tetrahedron, and its equation will be of the form

$$2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0.$$

The equation of the reciprocal of this is (Arts. 67, 79)

$$2fmn\alpha^2 + 2gnl\beta^2 + 2hlm\gamma^2 + 2fgh\delta^2$$
$$+ 2(fl - gm - hn)\ (l\beta\gamma + f\alpha\delta) + 2(gm - hn - fl)\ (m\gamma\alpha + g\beta\delta)$$
$$+ 2(hn - fl - gm)\ (n\alpha\beta + h\gamma\delta) = 0.$$

If now we write for $\alpha\sqrt{(fmn)}$, $\beta\sqrt{(gnl)}$, $\gamma\sqrt{(hlm)}$, $\delta\sqrt{(fgh)}$, x, y, z, w respectively, this equation becomes

$$x^2 + y^2 + z^2 + w^2 + \frac{fl - gm - hn}{\sqrt{(ghmn)}}(yz + xw)$$
$$+ \frac{gm - hn - fl}{\sqrt{(hfnl)}}(zx + yw) + \frac{hn - fl - gm}{\sqrt{(fglm)}}(xy + zw) = 0.$$

Now it is easy to see that these three coefficients are respectively $-2\cos A$, $-2\cos B$, $-2\cos C$, where A, B, C are the angles of a plane triangle whose sides are $\sqrt{(fl)}$, $\sqrt{(gm)}$, $\sqrt{(hn)}$. Hence, the general form of the equation of a quadric touching the four planes of reference is

$$x^2 + y^2 + z^2 + w^2 - 2p\ (yz + xw)\ - 2q\ (zx + yw) - 2r\ (xy + zw) = 0,$$

where p, q, r are the cosines of the angles of a plane triangle, or, in other words, are subject to the condition $1 - 2pqr = p^2 + q^2 + r^2$.

It may be seen otherwise that the surface whose equation has been written is actually touched by the four planes; for the condition just stated is the condition of the vanishing of the discriminant of the conic obtained by writing x, y, z, or $w = 0$, in the equation of the quadric. The section therefore

by each of the four planes being two real or imaginary lines, each of these planes is a tangent plane.

209. If V represents *a cone* we have $\Delta'=0$, and we proceed to examine the meaning in this case of Θ, Φ, Θ'. For simplicity we may take the origin as the vertex of V, or l', m', n', d' all $=0$. We have then

$$\Theta' = d\ (a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2),$$

or Θ' vanishes either if the cone break up into two planes, or if the vertex of the cone be on the surface U. Let the cone whose vertex is the origin and which circumscribes U, viz.

$$d(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) - (lx + my + nz)^2$$

be written

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}zx + 2\mathrm{h}xy = 0,$$

then Φ may be written

$$\mathrm{a}\ (b'c' - f'^2) + \mathrm{b}\ (c'a' - g'^2) + \mathrm{c}\ (a'b' - h'^2)$$
$$+ 2\mathrm{f}\ (g'h' - a'f') + 2\mathrm{g}\ (h'f' - b'g') + 2\mathrm{h}\ (f'g' - c'h').$$

Hence, by the theory of the invariants of plane conics (*Conics*, Art. 375) $\Phi = 0$ expresses the condition that it shall be possible to draw three tangent lines to U from the vertex of the cone V, which shall form a system self-conjugate with regard to V. In like manner

$$d\Theta = a'\ (\mathrm{bc} - \mathrm{f}^2) + b'\ (\mathrm{ca} - \mathrm{g}^2) + \&c.,$$

or Θ vanishes whenever three tangent planes to U can be drawn from the vertex of the cone V which shall form a system self-conjugate with regard to V. The discriminant of the cubic in λ will vanish when the cone V touches U.

When V represents *two planes*, both Δ' and Θ' vanish. Let the two planes be x and y, then V reduces to $2h'xy$, and Φ reduces to $h'^2(n^2 - cd)$, Φ will vanish therefore in this case when the intersection of the two planes touches U. We have $\Theta = 2h'H$ (see Art. 67), and its vanishing expresses the condition that the two planes should be conjugate with respect to U; or, in other words, that the pole of either with regard to U, should lie on the other. For (see Art. 79) the coordinates of the pole of the plane x are proportional to A, H, G, L, and the pole will therefore lie in the plane y when

$H=0$. The condition $\Theta^2=4\Delta\Phi$ is satisfied if either of the two planes touches U.

[Ex. If U and V are both cones interpret the meaning of the evanescence of one or more of the invariants Θ, Θ', Φ.]

210. The plane at infinity cuts any sphere in an imaginary circle the cone standing on which, and whose vertex is the origin, is $x^2+y^2+z^2=0$. Further, since this equation also represents an infinitely small sphere, any diameter is perpendicular to the conjugate plane (cf. Art. 139). If now we form the invariants of $x^2+y^2+z^2$, and the quadric given by the general equation, we get $\Theta=0$, or $A+B+C=0$, as the condition that the origin shall be a point whence three rectangular tangent planes can be drawn to the surface, and $\Phi=0$, or

$$ad-l^2+bd-m^2+cd-n^2=0,$$

as the condition that the origin shall be a point whence three rectangular tangent lines can be drawn to the surface.

In particular if the origin be the centre and therefore l, m, n all $=0$, and (the surface not being a cone) d not $=0$, the cubic is the same as that worked out (Art. 82). The condition $\Phi=0$ reduces to $a+b+c=0$, as *the condition that it shall be possible to draw systems of three rectangular asymptotic lines to the surface;* and the condition $\Theta=0$, gives

$$bc+ca+ab-f^2-g^2-h^2=0,$$

as the *condition that it shall be possible to draw systems of three rectangular asymptotic planes to the surface.*

These two kinds of hyperboloids answer to equilateral hyperbolas in the theory of plane curves (see Ex. 3, Art. 201); the former were called *equilateral* hyperboloids (Ex. 8, Art. 121). But *orthogonal* hyperboloids (Ex. 5*a*, Art. 121) are of a distinct kind, answering in a similar manner to circles in the theory of plane curves, and the relation among the coefficients can be found by investigating when the pole with regard to the cone $x^2+y^2+z^2=0$, of one of its planes of intersection with the general cone $(a, b, c, f, g, h)(xyz)^2=0$, will lie on the latter, which is parallel to the asymptotic cone of the general quadric.

Ex. Every equilateral hyperbola which passes through three fixed points passes through a fourth; what corresponds in the theory of quadrics? It will be seen that the truth of the plane theorem depends on the fact that the

condition that the general equation of a conic shall represent an equilateral hyperbola is linear in the coefficients. Thus, then, every rectangular hyperboloid (viz. hyperboloid fulfilling such a relation as $a + b + c = 0$) which passes through seven points passes through a fixed curve, and one which passes through six fixed points passes through two other fixed points. For the conditions that the surface shall pass through seven points together with the given relation enable us to determine all the coefficients of the quadric except one. Its equation therefore containing but one indeterminate is of the form $U + kV$ which passes through a fixed curve. And when six points are given the equation can be brought to the form $U + kV + lW$ which passes through eight fixed points.

211. Since any tangent plane to the cone $x^2+y^2+z^2$ is $xx'+yy'+zz'=0$ where $x'^2+y'^2+z'^2=0$, and since any parallel plane passes through the same line at infinity, we see that $\alpha^2+\beta^2+\gamma^2=0$ is the condition that the plane $\alpha x+\beta y+\gamma z+\delta$ shall pass through one of the tangent lines to the imaginary circle at infinity through which all spheres pass (Art. 139). And therefore $\alpha^2+\beta^2+\gamma^2=0$ may be said to be the *tangential equation of this circle.* [See Art. 144 (d).] The invariants formed with $\alpha^2+\beta^2+\gamma^2$ and the tangential equation of the surface are

$$\Theta = \Delta^2\,(a+b+c),\ \Phi = \Delta\,(bc-f^2+ca-g^2+ab-h^2),$$

the geometrical meaning of which has been stated in the last article.

The condition that two planes should be at right angles viz. $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$ (Art. 29), being the same as the condition that two planes should be conjugate with regard to $\alpha^2+\beta^2+\gamma^2$, we see that *two planes at right angles are conjugate with regard to the imaginary circle at infinity;* or, what is the same thing, their intersections with the plane infinity are conjugate in regard to the circle.*

212. In general, the *tangential equation of a curve in space expresses the condition that any plane should pass through one of the tangents of the curve.* For instance, the condition (Art. 80) that the intersection of the planes $\alpha x+\beta y+\gamma z+\delta w$,

* [This result evidently follows from the Projective Definition of angles when the Absolute is the quadric generated by tangent lines to the imaginary circle at infinity. Art. 144 (d).]

$\alpha'x+\beta'y+\gamma'z+\delta'w$ should touch a quadric, may be considered as the tangential equation of the conic in which the quadric is met by the plane $\alpha'x+\beta'y+\gamma'z+\delta'w$.

The reciprocal of a plane curve is a cone (Art. 123), and since an ordinary equation of the second degree whose discriminant vanishes represents a cone, so *a tangential equation of the second degree whose discriminant vanishes represents a plane conic.* From such a tangential equation

$$A\alpha^2+B\beta^2+\&c.=0,$$

we can derive the ordinary equations of the curve, by first forming the reciprocal of the given tangential equation according to the ordinary rules, $(BCD+\&c.)x^2+\&c.$, when we shall obtain a perfect square, viz. the square of the equation of the plane of the curve. And the conic is determined, by combining with this the equation

$$x^2(BC-F^2)+y^2(CA-G^2)+z^2(AB-H^2)$$
$$+2yz(GH-AF)+2zx(HF-BG)+2xy(FG-CH)=0,$$

which represents the cone joining the conic to the origin.

213. *To find the equation of the cone which touches a quadric U along the section made in it by any plane*

$$\alpha x+\beta y+\gamma z+\delta w=0.$$

The equation of any quadric touching U along this plane section being $kU+(\alpha x+\beta y+\gamma z+\delta w)^2=0$, it is required to determine k so that this shall represent a cone. We find in this case Φ, Θ', Δ' all $=0$. And if we denote by σ the quantity $A\alpha^2+B\beta^2+\&c.$ (Art. 79), the equation to determine k has three roots $=0$, the fourth root being given by the equation $k\Delta+\sigma=0$. The equation of the required cone is therefore $\sigma U=\Delta(\alpha x+\beta y+\gamma z+\delta w)^2$. When the given plane touches U, we have $\sigma=0$, Art. 79, and the cone reduces to the tangent plane itself, as evidently ought to be the case.

Under the problem of this article is included that of finding the *equation of the asymptotic cone to a quadric given by the general equation.*

214. The condition $\sigma=0$, that $\alpha x+\beta y+\gamma z+\delta w$ should touch U, is a *contravariant* (see *Conics*, Art. 380) of the third

order in the coefficients. If we substitute for each coefficient a, $a+\lambda a'$, &c., we shall get the condition that $\alpha x+\beta y+\gamma z+\delta w$ shall touch the surface $U+\lambda V$, a condition which will be of the form $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'=0$. [Hence three surfaces of the system touch any given plane. The points of contact are the vertices of the common self-conjugate triangle of the conics in which any two of the quadrics meet the plane, and the corresponding generators are the three pairs of lines through the intersections of the conics.] The functions σ, σ', τ, τ' each contain α, β, &c. in the second degree, and the coefficients of U and V in the third degree. In terms of these functions can be expressed the condition that the plane $\alpha x+\beta y+\gamma z+\delta w$ should have any permanent relation to the surfaces U, V; as for instance that it should cut them in sections u, v, connected by such permanent relations as can be expressed by relations between the coefficients of the discriminant of $u+\lambda v$. Thus if we form the discriminant with respect to λ of $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'$, we get the condition that $\alpha x+\beta y+\gamma z+\delta w$ should meet the surfaces in sections which touch; or, in other words, the *condition that this plane should pass through a tangent line of the curve of intersection of U and V*. This condition is

$$4(3\sigma'\tau-\tau'^2)(3\sigma\tau'-\tau^2)=(9\sigma\sigma'-\tau\tau')^2$$

and is of the eighth order in α, β, γ, δ, and of the sixth order in the coefficients of each of the surfaces. Thus, again, $\tau=0$ expresses the condition that the plane should cut the surfaces in two sections such that a triangle self-conjugate with respect to one can be inscribed in the other, &c.

[If the cubic for λ has three equal roots the plane cuts the quadrics in two sections which osculate, and therefore meets the curve of intersection of U and V in three coincident points. It is called the *osculating plane of the curve* (U, V) and is a tangent plane to the developable formed by tangent lines to the curve.]

The equation $\sigma=0$ may also be regarded as the tangential equation of the surface U; and, in like manner, $\tau=0$, $\tau'=0$ are tangential equations of quadrics having fixed relations to

U and V. Thus, from what we have just seen, $\tau=0$ is the envelope of a plane cutting the surface in two sections having to each other the relation just stated. And *the discriminant of* $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'$ *is the tangential equation of the curve of intersection of* U *and* V.

Or, again, $\sigma=0$ may be regarded as the equation of the surface reciprocal to U with regard to $x^2+y^2+z^2+w^2$ (Art. 127). And, in like manner, $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'$ is the equation of the surface reciprocal to $U+\lambda V$. Since if λ varies, $U+\lambda V$ denotes a series of quadrics passing through a common curve, the reciprocal system touches a common developable, which is the reciprocal of the curve UV. And the discriminant of $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'$ may be regarded at pleasure as the tangential equation of the curve UV, or as the equation of the reciprocal *developable*. This equation is, as was remarked above, of the eighth degree in the new variables, and of the sixth in the coefficients of each surface.

When $\Delta=0$, σ is the square of a linear function of $\alpha, \beta, \gamma, \delta$; and when the surface consists of two planes each first minor of Δ vanishes, and therefore in this case σ vanishes identically.

215. We can reciprocate the process employed in the last article. If $\sigma=0$, $\sigma'=0$ be the tangential equations of two quadrics, we can form the equation in ordinary coordinates answering to $\sigma+\lambda\sigma'$. This will be of the form

$$\Delta^2U+\lambda\Delta T+\lambda^2\Delta' T'+\lambda^3\Delta'^2V=0,$$

and will represent *a system of quadrics all touching a common developable, whose equation is found by forming the discriminant of the equation* last written. Thus, for example, using the canonical forms, let

$$U=ax^2+by^2+cz^2+dw^2,\quad V=a'x^2+b'y^2+c'z^2+d'w^2;$$

then $\sigma=A\alpha^2+B\beta^2+C\gamma^2+D\delta^2$, $\sigma'=A'\alpha^2+B'\beta^2+C'\gamma^2+D'\delta^2$, where $A=bcd$, $B=cda$, &c., and the reciprocal of $\sigma+\lambda\sigma'$ is
$\{BCDx^2+\&c.\}+\lambda\{(BCD'+CDB'+DBC')x^2+\&c.\}$
$+\lambda^2\{(B'C'D+C'D'B+D'B'C)x^2+\&c.\}+\lambda^3\{B'C'D'x^2+\&c.\}=0.$
Putting in the values for B, C, D, &c., we find

$$BCDx^2+\&c.=\Delta^2U,$$

while the coefficient of λ is

$$\Delta\{aa'(b'c'd+c'd'b+d'b'c)x^2+\text{\&c.}\}.$$

Just as all contravariants of the system σ, σ' can be expressed in terms of two fixed contravariants τ, τ' together with σ, σ', so all *covariants* of the system U, V can be expressed in terms of the two fixed covariants T, T' together with U, V and the invariants (Art. 200). Reciprocating what was stated in the last article we can see that the quadric T is the locus of a point whence cones circumscribing U and V are so related that three edges of one can be found, which form a self-conjugate system with regard to the second, and three tangent planes of the second which form a self-conjugate system with regard to the first.

If we please we may use instead of T and T' the quadric S, which is the locus of the poles with respect to V of all the tangent planes to U, and S' the locus of the poles with respect to U of all the tangent planes to V (see Ex. 12, Art. 121). By the help of the canonical form we can see what relations connect S and S' with T and T'. Thus we easily find

$$S=bcda'^2x^2+cdab'^2y^2+dabc'^2z^2+abcd'^2w^2.$$

But $T'=aa'(bcd'+cdb'+dbc')x^2+\text{\&c.}$

$=(bcda'+cdab'+dabc'+abcd')\ (a'x^2+\text{\&c.})-(bcda'^2x^2+\text{\&c.})$,

hence $T'=\Theta V-S$, and in like manner $T=\Theta' U-S'$. It appears thus that U, S', T have a common curve of intersection.

Ex. 1. To find the locus of a point whose polar planes with respect to U touch $U+\lambda V$. We have then in $\sigma+\lambda\tau+\lambda^2\tau'+\lambda^3\sigma'$ to substitute U_1, U_2, U_3, U_4 for α, β, γ, δ. The result is expressible in terms of the covariants by means of the canonical forms

$$U=x^2+y^2+z^2+w^2,\ V=ax^2+by^2+cz^2+dw^2.$$ For the result is

$$x^2+\text{\&c.}+\lambda\{(b+c+d)x^2+\text{\&c.}\}+\lambda^2\{(bc+cd+db)x^2+\text{\&c.}\}$$
$$+\lambda^3(bcdx^2+\text{\&c.})=0,$$

or $$\Delta U+\lambda(\Theta U-\Delta V)+\lambda^2(\Phi U-T')+\lambda^3(\Theta' U-T)=0.$$

In like manner the locus of points, whose polar planes with respect to V touch $U+\lambda V$, is

$$\overline{S}=\Theta V-T'+\lambda(\Phi V-T)+\lambda^2(\Theta' V-\Delta' U)+\lambda^3\Delta' V=0.$$

Ex. 2. To find the locus of a point whose polar planes with respect to U and V are a conjugate pair with regard to $U+\lambda V$. In the same manner that the condition that two points should be conjugate with respect to V is

$$ax'x''+by'y''+\text{\&c.}=0,$$

so the condition that two planes should be conjugate is $A\alpha\alpha'+B\beta\beta'+\text{\&c.}=0$.

Applying this to the case where α, β are U_1, U_2, &c., we get for the canonical form

$$ax^2 + \&c. + \lambda\{(b + c + d)\, ax^2 + \&c.\} + \lambda^2\{(bc + cd + db)\, ax^2 + \&c.\} + \lambda^3 abcd\,(x^2 + \&c.)$$

or
$$\Delta V + \lambda T' + \lambda^2 T + \lambda^3 \Delta' U = 0.$$

Ex. 3. To find the discriminant of T. *Ans.* $\Delta\Delta'\{\Theta'^2\Phi - \Delta'(\Theta\Theta' - \Delta\Delta')\}$.

216. What has been stated in the last article enables us to write down the *equation of the developable circumscribing two given quadrics U, V.** We have seen that its equation is the discriminant of the cubic $\varDelta^2 U + \lambda\varDelta T + \lambda^2\varDelta' T' + \lambda^3\varDelta'^2 V$, where if

$$U = ax^2 + by^2 + cz^2 + dw^2,\quad T = aa'(b'c'd + c'd'b + d'b'c)x^2 + \&c.$$

Clearing the discriminant of the factor $\varDelta^2\varDelta'^2$, the result is

$$27\,\varDelta^2\varDelta'^2U^2V^2 + 4\varDelta' UT'^3 + 4\varDelta VT^3 = T^2T'^2 + 18\varDelta\varDelta' TT'UV,$$

an equation of the *eighth degree* in the variables, and the tenth in the coefficients of each of the quadrics. By making $U = 0$, we see that the developable touches U along the curve UT, and that it meets U again in the curve of intersection of U with $T'^2 - 4\varDelta VT$. We shall presently see that the latter locus represents eight right lines, real or imaginary generators of the quadric U.

It is otherwise evident what is the curve of contact of the developable with U. For the point of contact with U of a common tangent plane to UV is the pole with regard to U of a tangent plane to V, and therefore is a point on the surface S'; and we have proved, in the last article, that the curves US', TU are the same.

The *section of the developable by one of the principal planes* (w) is most easily obtained by reverting to the process whence the equation was formed. The common tangent developable of $x^2 + y^2 + z^2 + w^2$, $ax^2 + by^2 + cz^2 + dw^2$ is the discriminant of

$$\frac{ax^2}{\lambda + a} + \frac{by^2}{\lambda + b} + \frac{cz^2}{\lambda + c} + \frac{dw^2}{\lambda + d} = 0.$$

* [This developable is the locus of lines joining the points of contact of common tangent planes; or of lines which are the limits of the intersection of two common tangent planes, when these planes approach towards coincidence.]

Hence, as in Art. 202, Ex. 2, if we make $w=0$, the discriminant will be

$$\left(\frac{ax^2}{a-d}+\frac{by^2}{b-d}+\frac{cz^2}{c-d}\right)^2,$$

multiplied by the discriminant of

$$\frac{ax^2}{\lambda+a}+\frac{by^2}{\lambda+b}+\frac{cz^2}{\lambda+c}.$$

In order to obtain the latter discriminant, differentiate with regard to λ, when we have

$$\frac{ax^2}{(\lambda+a)^2}+\frac{by^2}{(\lambda+b)^2}+\frac{cz^2}{(\lambda+c)^2}=0, \quad \frac{a^2x^2}{(\lambda+a)^2}+\frac{b^2y^2}{(\lambda+b)^2}+\frac{c^2z^2}{(\lambda+c)^2}=0,$$

whence $\dfrac{ax^2}{(\lambda+a)^2}=b-c,\ \dfrac{by^2}{(\lambda+b)^2}=c-a,\ \dfrac{cz^2}{(\lambda+c)^2}=a-b$;

and, substituting in the given equation, the result is

$$x\sqrt{\{a(b-c)\}}\pm y\sqrt{\{b(c-a)\}}\pm z\sqrt{\{c(a-b)\}}=0.$$

The section therefore is a conic counted twice and four right lines. These lines touch the conic and intersect on the edges of the tetrahedron. They are the common tangents of the sections of U and V by the principal planes.

[The four conics correspond to the four values of λ for which $\lambda\sigma+\sigma'$ represents conics. By the method of Art. 136, we find that these four conics lie in the faces of the self-conjugate tetrahedron, just as reciprocally the vertices of the cones of the system $\lambda U+V$ are the vertices of the self-conjugate tetrahedron.

The circum-developable is generated by lines joining the points of contact of common tangent planes to any two of these conics (since they are limiting cases of two quadrics of the system $\sigma+\lambda\sigma'$). It is easy to see geometrically that the conics are double lines; for let the tangent plane at a point P on one of the conics meet the line of intersection of the planes of the conics in T; from T draw tangents TQ, TQ' to the second conic; then the lines PQ, PQ' are generators of the developable. Thus through any point on either conic two generators can be drawn.

Further let u, v be the sections of quadrics U, V of the system by a principal plane. The plane joining a common tangent line of u and v to the opposite vertex of the tetrahedron is a tangent plane to U and V at the points where the common tangent line meets u and v; hence this line is a generator of the developable. It evidently touches the double conic in the plane, since this conic is to be regarded as one of the conics in which the plane meets a quadric of the system.

Ex. 1. Express the coordinates of the point on the developable by means of two parameters. We find

$$x^2 = \frac{(a-p)^2(a-q)}{a(a-b)(a-c)(a-d)}, \text{ \&c.}$$

The sections by the principal planes can hence be found (cf. Art. 206).

Ex. 2. The "cuspidal edge" or the "edge of regression" on the developable is the curve locus of points for which the cubic in λ has three equal roots. Its coordinates may be expressed by a parameter p,

$$x^2 = \frac{(p-a)^3}{a(a-b)(a-c)(a-d)}, \text{ \&c.}$$

Prove that it meets a principal plane at the points where the four right lines (see above) touch the double conic.

Ex. 3. The coordinates of a tangent plane to the developable in terms of a single parameter p may be written

$$\alpha^2 = \frac{a(p+a)}{(a-b)(a-c)(a-d)}, \text{ \&c.}$$

Ex. 4. The developable meets the principal planes in a double conic. Taking the points where this meets the sections of U by the same planes we get sixteen points. Prove that these lie by fours in eight generators of U.

Ex. 5. Prove that the cuspidal edge is a curve of the twelfth degree.]

217. *To find the condition that a given line should pass through the curve of intersection of two quadrics U and V.*

Suppose that we have found, by Arts. 80, &c., the condition, $\Psi = 0$, that the line should touch U, and that we substitute in it for each coefficient a, $a + \lambda a'$, the condition becomes $\Psi + \lambda\Psi_1 + \lambda^2\Psi' = 0$; and if the line have any arbitrary position, we can by solving this quadratic for λ, determine two surfaces passing through the curve of intersection UV and touching the given line. But if the line itself pass through UV, then it is easy to see that these two surfaces must coincide, for the line cannot, in general, be touched by a surface of the system anywhere but in the point where it meets UV. The condition therefore which we are seeking is $\Psi_1^2 = 4\Psi\Psi'$. It is of the second order in the coefficients of each of the surfaces and of the fourth in the coefficients of each of the planes determining the right line: these (see Art. 80) enter through the combinations $\alpha\beta' - \alpha'\beta$, &c., viz. the equation contains, and that in the fourth degree, the six coordinates of the line of intersection of the two planes.

In the case where the two quadrics are $ax^2 + by^2 + cz^2 + dw^2$, $a'x^2 + b'y^2 + c'z^2 + d'w^2$, and the right line is $\alpha x + \beta y + \gamma z + \delta w$, $\alpha'x + \beta'y + \gamma'z + \delta'w$, the quantity Ψ is (see Art. 80)

$\Sigma ab\ (\gamma\delta' - \gamma'\delta)^2$, by which notation we mean to express the sum of the six terms of like form, such as $cd\ (\alpha\beta' - \alpha'\beta)^2$, &c. When the line is expressed by its ray coordinates (Art. 53*a*) the relation which holds for contact is

$$bcp^2 + caq^2 + abr^2 + ads^2 + bdt^2 + cdu^2 = 0,$$

which is satisfied by each of the *complex of lines* which touch the quadric U (see Art. 80*f*). Then Ψ_1 is $\Sigma(bc' + b'c)p^2$, and its vanishing is the relation for the complex of all lines which are cut harmonically by the quadrics U and V, as it is easily seen that $\Psi_1 = U'V'' + U''V' - 2PQ$ in the notation of Art. 75. Also $\Psi_1{}^2 - 4\Psi\Psi'$ is

$$\Sigma(bc')^2p^4 + 2\Sigma(bc')\,(ac')\,p^2q^2 + 2\Sigma\{(ab')\,(cd') + (ac')\,(bd')\}\,p^2s^2,$$

and vanishes for the complex of right lines intersecting the common curve.

218. *To find the equation of the developable generated by the tangent lines of the curve common to U and V.*

If we consider any point on any tangent to this curve, the polar plane of this point with regard to either U or V passes evidently through the point of contact of the tangent on which it lies. The intersection therefore of the two polar planes meets the curve UV. We find thus the equation of the developable required, by substituting in the condition of the last article, for α, β, &c., α', β', &c., the differential coefficients U_1, U_2, &c., V_1, V_2, &c. This developable will be of the *eighth degree* in the variables and of the sixth in the coefficients of each surface. When we use the canonical form of the quadrics, it then easily appears that the result is

$$\Sigma(ab')^2\,(cd')^4z^4w^4 + 2\Sigma(ab')\,(ac')\,(cd')^2\,(bd')^2y^2z^2w^4 + 2x^2y^2z^2w^2$$
$$\times\{(ab')\,(cd') - (ad')\,(bc')\}\,\{(ad')\,(bc') - (bd')\,(ca')\}\,\{(bd')\,(ca') - (ab')\,(cd')\}.$$

When we make in the above equation $w = 0$ we obtain a perfect square, hence *each of the four planes, x, y, z, w meets the developable in a plane curve of the fourth degree which is a double line on the surface.**

* See *Cambridge and Dublin Mathematical Journal*, Vol. III., p. 171, where, though only the geometrical proof is given, I had arrived at the result by

[This may be proved geometrically by showing that the tangent line at any point M on the curve UV is met by another tangent line where it cuts the plane $w = 0$. If A is the pole of the plane $w = 0$, the line AM, being a generator of the quadric cone which joins the curve to A, cuts the curve again at N, and the tangents at M and N lie in the plane touching this cone along AM. They thus intersect, and their intersection lies in the polar plane of A with respect to any quadric through the curve.]

By the help of the canonical form the previous result can be expressed in terms of the covariant quadrics when the developable is found to be

$$4(\Theta UV - T'U - \Delta V^2)(\Theta' UV - TV - \Delta' U^2) = (\Phi UV - TU - T'V)^2.$$

The curve UV is manifestly a double line on the locus represented by this equation, as we otherwise know it to be, and the locus meets U again in the locus of the eighth order determined by the intersection of U with $T'^2 - 4\Delta TV$. This is the same locus as that found in Art. 216, and we shall prove that it represents eight straight lines.

It is proved, as at *Higher Plane Curves*, Art. 51 (see also Art. 110 of this volume) that when the equation of a surface is $U^2\phi + UV\psi + V^2\chi = 0$, then UV is a double line on the surface, the two tangent planes at any point of it being given by the equation $u^2\phi' + uv\psi' + v^2\chi' = 0$, where u, v are the tangent planes at that point to U and V, and ϕ' is the result of substituting in ϕ the coordinates of this point, &c. Applying this to the above equation it is immediately found that the two tangent planes are given by the equation $(Tu - T'v)^2 = 0$, where in T', T' the coordinates of the point are supposed to be substituted. Thus *the two tangent planes at every point of the double curve coincide, and the curve is accordingly called a cuspidal curve on the surface.*

219. We can show geometrically (as was stated Art. 216) that a generator of the quadric U at each of the eight points of intersection of the three surfaces U, V, S (or U, V, T') is also a generator of the developable, and that therefore these eight lines form the locus of the eighth order, U, $T'^2 - 4\Delta TV$. For the surface S being the locus of the poles with regard to V of the tangent planes to U, the tangent plane to V at one of the eight points in question is also a tangent plane to U

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actual formation of the equation of the developable. See *ibid.*, Vol. II., p. 68. The equations were also worked out by Cayley, *ibid.*, Vol. V., pp. 50, 51.

and therefore passes through one of the generators to U at the same point. This generator is therefore the line of intersection of the tangent planes to U and V, and therefore is a generator of the developable in question.

[The generator of U, being a line joining the points of contact of a common tangent plane to U and V, is also a generator of the circum-developable.

It may be shown that *these eight generators lie by twos in four planes through any vertex of the tetrahedron*, and reciprocally, they *intersect by twos in four points on any face of the tetrahedron* (cf. Art. 216, Ex. 4). In fact (taking the latter theorem) if a generator of U is a generator of the circum-developable it meets all the four double conics (Art. 216) and therefore passes through the intersection of U with these conics; and the former theorem is proved reciprocally.

Ex. 1. State and prove the reciprocals of the theorems and examples in Art. 216.

Ex. 2. Prove that three and only three quadrics of the system $U + \lambda V = 0$ can be drawn to touch a given plane, and that the lines joining the points of contact to a vertex of the common self-conjugate tetrahedron are conjugate diameters of the corresponding cone of the system.

State and prove the reciprocal theorem for the system $\sigma + \lambda\sigma' = 0$.]

220. The calculation in Art. 218 may also be made as follows: When we write in the determinant of Art. 80 for a, $a + \lambda a'$, &c., and for α, β, &c. U_1, U_2, &c., for α', β', &c. V_1, V_2, &c., we can reduce it by subtracting from the first column the sum of the third multiplied by x, of the fourth, fifth, and sixth multiplied respectively by y, z, and w, and then, removing the terms $-\lambda V_1$, &c. in the first column by means of V_1, &c. in the second; when we deal similarly with the rows, the determinant becomes

$$(U+\lambda V)\overline{S} - V^2(\Delta + \lambda\Theta + \lambda^2\Phi + \lambda^3\Theta' + \lambda^4\Delta'),$$

where $-\overline{S}$ is the value of the determinant of Art. 79, when a, &c. are replaced by $a+\lambda a'$, &c. and α, &c. by V_1, &c. But the last result of Ex. 1, Art. 215, determined the value of $\overline{S}$. Putting in that value we find, as it should be, that λ occurs in no higher power than the second, and the determinant becomes

$$(\Theta UV - T'U - \Delta V^2) + \lambda(\Phi UV - TU - T'V) + \lambda^2(\Theta' UV - TV - \Delta' U^2) = 0.$$

Thus then we see that $\Theta UV = T'U + \Delta V^2$ is the condition

that the intersection of the two polar planes should touch U; while $\Phi UV = TU + T'V$ is the condition that it should be cut harmonically by the surfaces U, V; and again the equation of the developable is

$$4(\Theta UV - T'U - \Delta V^2)(\Theta' UV - TV - \Delta' U^2) = (\Phi UV - TU - T'V)^2.$$

220*a*. The equation of this developable has been otherwise derived by W. R. Roberts as follows: When the line whose ray coordinates are p, q, r, s, t, u is a generator of

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

we have (Art. 80*g*)

$$\begin{aligned} 0 &= \phantom{cp^2 + {}} cq^2 + br^2 + ds^2 , \\ 0 &= cp^2 \phantom{{}+ cq^2} + ar^2 \phantom{{}+ ds^2} + dt^2 , \\ 0 &= bp^2 + aq^2 \phantom{{}+ ar^2 + dt^2} + du^2, \\ 0 &= \phantom{bp^2 + aq^2 + {}} as^2 + bt^2 + cu^2, \end{aligned}$$

which are equivalent to the four equations

$$p^2 = s^2\frac{ad}{bc},\ q^2 = t^2\frac{bd}{ca},\ r^2 = u^2\frac{cd}{ab},\ as^2 + bt^2 + cu^2 = 0.$$

Now a generator of any one of the system of quadrics through the curve common to U and V is a line which meets that curve in two points; hence the line whose coordinates are related as follows:

$$p^2 = s^2\frac{(a+\lambda a')}{(b+\lambda b')}\frac{(d+\lambda d')}{(c+\lambda c')},\ q^2 = t^2\frac{(b+\lambda b')}{(c+\lambda c')}\frac{(d+\lambda d')}{(a+\lambda a')},$$

$$r^2 = u^2\frac{(c+\lambda c')}{(a+\lambda a')}\frac{(d+\lambda d')}{(b+\lambda b')},\ (a+\lambda a')s^2 + (b+\lambda b')t^2 + (c+\lambda c')u^2 = 0,$$

is a generator of $U + \lambda V$ and a chord of the curve of intersection of

$$\begin{aligned} U &= ax^2 + by^2 + cz^2 + dw^2 = 0, \\ V &= a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0. \end{aligned}$$

220*b*. Again, when a line touches the curve UV, it touches both U and V, hence, in this case

$$\begin{aligned} bcp^2 + caq^2 + abr^2 + ads^2 + bdt^2 + cdu^2 &= 0, \\ b'c'p^2 + c'a'q^2 + a'b'r^2 + a'd's^2 + b'd't^2 + c'd'u^2 &= 0, \end{aligned}$$

therefore by the fourth relation in last article

$$(bcd' + \lambda b'c'd)p^2 + (cad' + \lambda c'a'd)q^2 + (abd' + \lambda a'b'd)r^2 = 0.$$

or, replacing p^2, q^2, r^2, by their values in s^2, t^2, u^2,

$$(bcd' + \lambda b'c'd)(a+\lambda a')^2 s^2 + (cad' + \lambda c'a'd)(b+\lambda b')^2 t^2 + (abd' + \lambda a'b'd)(c+\lambda c')^2 u^2 = 0.$$

Solving between this and

$$(a+\lambda a')\, s^2 + (b+\lambda b')\, t^2 + (c+\lambda c')\, u^2 = 0,$$

we get s^2, t^2, u^2, and accordingly also p^2, q^2, r^2.

Omitting a common factor, the results may be written

$$\begin{aligned} p^2 &= (bc')\,(ad')\,(a+\lambda a')\,(d+\lambda d'),\\ q^2 &= (ca')\,(bd')\,(b+\lambda b')\,(d+\lambda d'),\\ r^2 &= (ab')\,(cd')\,(c+\lambda c')\,(d+\lambda d'),\\ s^2 &= (bc')\,(ad')\,(b+\lambda b')\,(c+\lambda c'),\\ t^2 &= (ca')\,(bd')\,(c+\lambda c')\,(a+\lambda a'),\\ u^2 &= (ab')\,(cd')\,(a+\lambda a')\,(b+\lambda b'), \end{aligned}$$

and evidently admit of $ps + qt + ru = 0$ being identically satisfied.

220*c*. From these expressions in the parameter λ, for the coordinates of any generator, the equation of the developable may be found in ordinary coordinates by the usual method. For any point on the line we must have, for instance,

$$px + qy + rz = 0,$$

but we have also $U + \lambda V = 0$, hence the surface is

$$x\,\{(bc')\,(ad')\,(aV - a'U)\}^{\frac{1}{2}} + y\,\{(ca')\,(bd')\,(bV - b'U)\}^{\frac{1}{2}} + z\,\{(ab')\,(cd')\,(cV - c'U)\}^{\frac{1}{2}} = 0,$$

and the *section by the plane $z = 0$ is seen at once to be a double curve which is a trinodal quartic;* and similarly for the other planes of reference. The three nodes are the vertices of the tetrahedron of reference. Again, this equation of the surface evidently, on rationalisation, becomes of the form

$$U^2\phi + UV\psi + V^2\chi,$$

whence UV is a double line on it; also, making $U = 0$, $\sqrt{V}$ becomes a factor, and the eight right lines forming the remaining intersection with U are at once found.

220*d*. If the line pqr, &c. be contained in the plane $\alpha x + \beta y + \gamma z + \delta w = 0$ its coordinates satisfy $\alpha s + \beta t + \gamma u = 0$ &c. (Art. 53*b*). If the consecutive line also lie in this plane

$$a\frac{ds}{d\lambda}+\beta\frac{dt}{d\lambda}+\gamma\frac{du}{d\lambda}=0.$$

By these, determining a, β, γ, it is seen that the following are symmetrical expressions for the *coordinates of the plane of two consecutive generators of the developable, i.e. of two consecutive tangents to the common curve UV*, (the *osculating plane* of the curve at the point) omitting a common factor.

$$\begin{aligned}
a^2\ (ab')\ (ac')\ (ad') &= (a+\lambda a')^3,\\
\beta^2\ (bc')\ (bd')\ (ba') &= (b+\lambda b')^3,\\
\gamma^2\ (cd')\ (ca')\ (cb') &= (c+\lambda c')^3,\\
\delta^2\ (da')\ (db')\ (dc') &= (d+\lambda d')^3,
\end{aligned}$$

also the expressions

$$\begin{aligned}
x^2\ (ab')\ (ac')\ (ad') &= a+\mu a',\\
y^2\ (bc')\ (bd')\ (ba') &= b+\mu b',\\
z^2\ (cd')\ (ca')\ (cb') &= c+\mu c',\\
w^2\ (da')\ (db')\ (dc') &= d+\mu d',
\end{aligned}$$

are easily seen to be those for the *coordinates of any point on the curve UV.*

[See examples, Art. 216, for reciprocal expressions. Notice that the osculating plane in one system corresponds to a point on the cuspidal edge in the reciprocal system. The former is a plane containing two consecutive generators, the latter is a point of intersection of two such generators.]

221. The equation $ax^2+by^2+cz^2+\lambda\ (x^2+y^2+z^2)=1$ denotes (Art. 104) a system of concentric quadrics having common planes of circular section. And the form of the equation shows that the system in question has common the imaginary curve in which the point sphere $x^2+y^2+z^2$ meets any quadric of the system. Again, since the tangential equation of the system of confocal quadrics

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1,$$

is
$$a\alpha^2+b\beta^2+c\gamma^2+\lambda\ (\alpha^2+\beta^2+\gamma^2)=1,$$

it follows reciprocally that *a system of confocal quadrics is touched by a common imaginary developable* (see Art. 146); namely, that enveloped by the tangent planes drawn to any surface of the system, through the tangent lines to the imaginary circle at infinity. The equation of this developable

is found by forming the discriminant with regard to λ of the equation of the system of quadrics. If we write $b-c=p$, $c-a=q$, $a-b=r$, the equation is

$$\begin{aligned}(x^2+y^2+z^2)^2(p^2x^4+q^2y^4+r^2z^4-2qry^2z^2-2rpz^2x^2-2pqx^2y^2)\\ +2p^2(q-r)x^6+2q^2(r-p)y^6+2r^2(p-q)z^6\\ +2p(pr-3q^2)x^4y^2-2q(qr-3p^2)x^2y^4-2p(pq-3r^2)x^4z^2\\ +2r(qr-3p^2)x^2z^4+2q(qp-3r^2)y^4z^2-2r(rp-3q^2)z^4y^2\\ +2\,(p-q)\,(q-r)\,(r-p)x^2y^2z^2+(p^4-6p^2qr)x^4\\ +(q^4-6q^2pr)y^4+(r^4-6r^2pq)z^4+2pq\,(pq-3r^2)x^2y^2\\ +2qr(qr-3p^2)y^2z^2+2rp(rp-3q^2)z^2x^2+2p^2qr(r-q)x^2\\ +2q^2rp\,(p-r)y^2+2r^2pq\,(q-p)z^2+p^2q^2r^2=0.\end{aligned}$$

It may be deduced from this equation, or as in Art. 202, that *the focal conics, and the imaginary circle at infinity, are double lines on the surface.* [They are the conics in the principal planes (Art. 216), one of these being the plane at infinity.]

222. In like manner, if $\sigma=0$ be the tangential equation of a quadric, and if we form the reciprocal $\sigma+\lambda\,(\alpha^2+\beta^2+\gamma^2)$, we get

$$\begin{aligned}\Delta^2U+\lambda\Delta[\{a(b+c)-g^2-h^2\}\,x^2+\{b(c+a)-h^2-f^2\}\,y^2\\ +\{c(a+b)-f^2-g^2\}\,z^2+\{d(a+b+c)-l^2-m^2-n^2\}\\ +2yz(af-gh)+2zx(bg-hf)+2xy(ch-fg)\\ +2x\,\{(b+c)l-hm-gn\}+2y\,\{(c+a)m-fn-hl\}\\ +2z\,\{(a+b)n-gl-fm\}]\\ +\lambda^2\{D(x^2+y^2+z^2)+A+B+C-2Lx-2My-2Nz\}\\ +\lambda^3=0.\end{aligned}$$

This is the *equation of a series of confocal surfaces*, and its discriminant with respect to λ will represent the developable considered in the last article. If we write the coefficients of λ and λ^2 respectively T and T', then $T=0$ denotes the locus of points whence three rectangular lines can be drawn to touch the given quadric, and $T'=0$ the locus of points whence three rectangular tangent planes can be drawn to the same quadric.

If the *paraboloid* $\dfrac{x^2}{a}+\dfrac{y^2}{b}+2z$ be treated in the same

way, we obtain, as the equation of a system of confocal surfaces,

$$(bx^2 + ay^2 + 2abz) + \lambda \{x^2 + y^2 + 2(a+b)z - ab\} + \lambda^2 \{2z - (a+b)\} - \lambda^3 = 0,$$

and the developable which they all touch is, if we write $a - b = r$,

$$\begin{aligned}
&4(x^2+y^2)^2 (x^2+y^2+z^2) + 16rz(x^2+y^2+z^2)\ (x^2-y^2) \\
&\quad + 4z(x^2+y^2)\ (ax^2+by^2) + 16r^2z^4 + 32r^2z^2(x^2+y^2) \\
&\quad + 24r\ (bx^2+ay^2)z^2 + (ax^2+by^2)^2 + 8r(bx^2+ay^2)\ (x^2-y^2) \\
&\quad + 12r^2x^2y^2 + 16(a+b)r^2z(x^2+y^2+z^2) - 12r^2z(ax^2+by^2) \\
&\quad + 12rabz(x^2-y^2) + 4r^2z^2(a^2+4ab+b^2) + 4r^2(b^2x^2+a^2y^2) \\
&\quad + 2abr(ax^2-by^2) + 4r^2ab(a+b)z + a^2b^2r^2 = 0.
\end{aligned}$$

The locus of intersection of three rectangular tangent planes to the paraboloid is the plane $2z = a + b$, and of three rectangular tangent lines is the paraboloid of revolution

$$x^2 + y^2 + 2(a+b)z = ab.$$

223. We shall now show that *several properties of confocal surfaces are particular cases of properties of systems inscribed in a common developable.* It will be rather more convenient to state first the reciprocal properties of systems having a common curve.

Since the condition that a quadric should touch a plane (Art. 79) involves the coefficients in the third degree, it follows that of a system of quadrics passing through a common curve, three can be drawn to touch a given plane, and reciprocally, that of a system inscribed in the same developable, three can be described through a given point. It is obvious that in the former case one can be described through a given point, and in the latter, one to touch a given plane. In either case, two can be described to touch a given line; for the condition that a quadric should touch a right line (Art. 80) involves the coefficients of the quadric in the second degree.

It is also evident geometrically, that only three quadrics of a system having a common curve can be drawn to touch a given plane. For this plane meets the common curve in

four points, through which the section by that plane of every surface of the system must pass. Now, since a tangent plane meets a quadric in two right lines, real or imaginary, (Art. 107) these right lines in this case can be only some one of the *three* pairs of right lines which can be drawn through the four points. The points of contact, which are the points where the lines of each pair intersect, are (*Conics*, Art. 146, Ex. 1) each the pole of the line joining the other two with regard to any conic passing through the four points. Hence (Art. 71) if the vertices of one of the four cones of the system be joined to the three points, the joining lines are conjugate diameters of this cone.

224. Now let there be a system of quadrics of the form $S+\lambda\ (x^2+y^2+z^2)$; since $x^2+y^2+z^2$ is a cone, the origin is one of the four vertices of cones of the system. And since $x^2+y^2+z^2$ is an infinitely small sphere, any three conjugate diameters are at right angles, and we conclude that three surfaces of the system can be drawn to touch any plane, and that the lines joining the three points of contact to the origin are at right angles to each other. Moreover as a system of concentric and confocal quadrics is reciprocal to a system of the form $S+\lambda\ (x^2+y^2+z^2)$, we infer that three confocal quadrics can be drawn through any point and that they cut at right angles.

Again (Art. 132) the polar planes of any point with regard to any system of the form $S+\lambda\ (x^2+y^2+z^2)$ pass through a right line, the plane joining which to the origin is perpendicular to the line joining the given point to the origin; as is evident from considering the particular surface of the system $x^2+y^2+z^2$. Reciprocally then the locus of the poles of a given plane with regard to a system of confocals is a line perpendicular to that plane.

[Ex. Deduce properties of confocals from the theorems of Arts. 133 and 134.]

225. We have seen that $\sigma+\lambda\ (\alpha^2+\beta^2+\gamma^2)$ is the tangential equation of a system of confocals: and when the discriminant

of this equation vanishes it represents one of the focal conics We can therefore *find the tangential equation of the focal conics of a given surface* by determining λ from the equation

$$D\lambda^2+(bc+ca+ab-f^2-g^2-h^2)\,\Delta\lambda^2+(a+b+c)\,\Delta^2\lambda+\Delta^3=0.$$

Thus, let the surface be

$$7x^2+6y^2+5z^2-4yz-4xy+10x+4y+6z+4=0,$$

we have $\Delta=-972$, and the cubic is

$$162\lambda^3+99\Delta\lambda^2+18\Delta^2\lambda+\Delta^3=0,$$

whose factors are $3\lambda+\Delta$, $6\lambda+\Delta$, $9\lambda+\Delta$, whence $\lambda=108$, 162, or 324.

The tangential equation of the given surface divided by 6 is

$$\alpha^2-8\beta^2-11\gamma^2+27\delta^2+26\beta\gamma+46\gamma\alpha+34\alpha\beta-54\alpha\delta-54\beta\delta-54\gamma\delta=0.$$

Thus then the tangential equations of the three focal conics are obtained by altering the first three terms of the equation last written into

$$19\alpha^2+10\beta^2+7\gamma^2,\ 28\alpha^2+19\beta^2+16\gamma^2,\ 55\alpha^2+46\beta^2+43\gamma^2,$$

respectively. Their ordinary equations are found, as in Art. 212, to be the intersections of

$$2x-2y+z+w,\ 11x^2+44y^2+11z^2-32yz+2zx-40xy;$$

$$2x+y-2z+w,\ 5x^2-3y^2+9z^2+2yz-16zx+2xy;$$

$$x+2y+2z+5w,\ 67x^2+68y^2+83z^2-24yz-62zx-32xy.$$

[Ex. Point out the connexion between the cubic of Art. 83 and the cubic of this Art.]

226. In order to find *in quadriplanar coordinates* the tangential equation of a surface confocal to a given one, it is necessary to find the equivalent in quadriplanar coordinates to the equation $\alpha^2+\beta^2+\gamma^2=0$.* It is evident that if x, y, z, w represent any four planes, and if their equations referred to any three rectangular axes be $X\cos A+Y\cos B+Z\cos C=p$, &c., then the coefficient of X in $\alpha x+\beta y+\gamma z+\delta w$ is

$$\alpha\cos A+\beta\cos A'+\gamma\cos A''+\delta\cos A''',$$

and the sum of the squares of the coefficients of X, Y, Z is

* This condition evidently expresses that the length is infinite of the perpendicular let fall from any point on any of the planes which satisfy the equation.

$$\alpha^2+\beta^2+\gamma^2+\delta^2-2\beta\gamma\cos(yz)-2\gamma\alpha\cos(zx)-2\alpha\beta\cos(xy)$$
$$-2\alpha\delta\cos(xw)-2\beta\delta\cos(yw)-2\gamma\delta\cos(zw),$$
where (yz) denotes the angle between the planes y, z &c. This quantity equated to nothing is *the tangential equation of the imaginary circle at infinity.** The processes of the last articles then can be repeated by substituting the quantity just written for $\alpha^2+\beta^2+\gamma^2$. We thus find, without difficulty, the condition that the general equation in quadriplanar coordinates should represent a paraboloid, or either class of rectangular hyperboloid; the equations of the loci of points whence systems of three tangent planes or tangent lines are at right angles; the equations of the focal conics, &c.

227. We have seen (Art. 211) that the condition in rectangular coordinates $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$, that the planes $\alpha x+$ &c., $\alpha' x+$ &c. should be at right angles, expresses that the planes should be conjugate with respect to the imaginary circle at infinity. It follows that *the condition of perpendicularity in quadriplanar coordinates is*

$$\alpha'\{\alpha-\beta\cos(xy)-\gamma\cos(xz)-\delta\cos(xw)\}$$
$$+\beta'\{-\alpha\cos(xy)+\beta-\gamma\cos(yz)-\delta\cos(yw)\}+\&c.=0.$$

Any theorems concerning perpendiculars may be generalized projectively by substituting any fixed conic for the imaginary circle at infinity; and thus, instead of a perpendicular line and plane, we get a line and plane which meet the plane of the fixed conic in a point and line which are pole and polar with respect to that conic (see *Conics*, Art. 356).

The theorems may be extended further (see *Conics*, Art. 385) by substituting for the fixed conic a fixed quadric (the "Absolute,") when instead of a line perpendicular to a plane, we should have a line passing through the pole of the plane with regard to the fixed quadric.†

Ex. Any tangent plane to a sphere is perpendicular to the corresponding radius.	Any plane section of a quadric is met in a conjugate line and point, by any tangent plane and the line joining its point of contact to the pole of the plane of section.

* [The Euclidean "Absolute": see Art. 144 (*d*).] † [See Art. 144 (*d*).]

228. The tangential equation of a sphere, in rectangular coordinates, is written down at once by expressing that the distance of the centre from any tangent plane is constant. The equation is therefore

$$(\alpha x' + \beta y' + \gamma z' + \delta)^2 = r^2\ (\alpha^2 + \beta^2 + \gamma^2).$$

If then x', y', z', w' be the coordinates of the centre of a sphere, *the tangential equation of the sphere in quadriplanar coordinates* must be

$$(\alpha x' + \beta y' + \gamma z' + \delta w')^2 = r^2\{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\alpha\beta \cos (xy) - \&c.\}.$$

If the sphere touch the four planes x, y, z, w, the coefficients of α^2, β^2, γ^2, δ^2 must vanish, and the tangential equation of such a sphere must therefore be

$$(\alpha \pm \beta \pm \gamma \pm \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\alpha\beta \cos (xy) - \&c.$$

There are therefore eight spheres which touch the faces of a tetrahedron. Taking all positive signs, we get the *tangential equation of the inscribed sphere*

$$\beta\gamma \cos^2\tfrac{1}{2}\ (yz) + \gamma\alpha \cos^2\tfrac{1}{2}\ (zx) + \alpha\beta \cos^2\tfrac{1}{2}\ (xy)$$
$$+ \alpha\delta \cos^2\tfrac{1}{2}\ (xw) + \beta\delta \cos^2\tfrac{1}{2}\ (yw) + \gamma\delta \cos^2\tfrac{1}{2}\ (zw) = 0.$$

The corresponding quadriplanar equation is obtained from this as in Art. 208.

229. *The equation of the sphere circumscribing a tetrahedron* may be most simply obtained as follows: Let the four perpendiculars on each face from the opposite vertex be x_0, y_0, z_0, w_0. Now the equation *in plano* of the circle circumscribing any triangle abc may be written in the form

$$\frac{(bc)^2 yz}{y_0 z_0} + \frac{(ca)^2 zx}{z_0 x_0} + \frac{(ab)^2 xy}{x_0 y_0} = 0,$$

where x, x_0, &c. denote perpendiculars on the sides of a triangle the lengths of which are (bc), &c. But it is evident that for any point in the face w, the ratio $x : x_0$ is the same whether x and x_0 denote perpendiculars on the plane x or on the line xw. We are thus led to the equation required, viz.

$$\frac{(bc)^2 yz}{y_0 z_0} + \frac{(ca)^2 zx}{z_0 x_0} + \frac{(ab)^2 xy}{x_0 y_0} + \frac{(ad)^2 xw}{x_0 w_0} + \frac{(bd)^2 yw}{y_0 w_0} + \frac{(cd)^2 zw}{z_0 w_0} = 0.$$

For this is a quadric whose intersection with each of the four faces is the circle circumscribing the triangle of which that

face consists. If this equation be reduced to rectangular coordinates it will be found that the coefficients of x^2, y^2, z^2 are each $= -1$. Hence if we substitute the coordinates of any point, we get $-$ the square of the tangent from that point to the sphere.

Cor. The square of the distance between the centres of the inscribed and circumscribing spheres is

$$D^2 = R^2 - r^2\left\{\frac{(bc)^2}{y_0z_0} + \frac{(ca)^2}{z_0x_0} + \frac{(ab)^2}{x_0y_0} + \frac{(ad)^2}{x_0w_0} + \frac{(bd)^2}{y_0w_0} + \frac{(cd)^2}{z_0w_0}\right\}.$$

230. The equation of any other sphere can only differ from the preceding by terms of the first degree, which must be of the form $(\alpha x + \beta y + \gamma z + \delta w)\left(\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} + \frac{w}{w_0}\right)$, the second factor denoting the plane at infinity (Art. 41). If then we add to the equation of the last article the product of these two factors, identify with the general equation of the second degree and eliminate the indeterminate constants, we obtain the *conditions that the general equation of the second degree in quadriplanar coordinates* $\mathrm{a}x^2 + \mathrm{b}y^2 + \&c.$ *may represent a sphere*, viz.

$$\frac{\mathrm{b}y_0^2 + \mathrm{c}z_0^2 - 2\mathrm{f}y_0z_0}{(bc)^2} = \frac{\mathrm{c}z_0^2 + \mathrm{a}x_0^2 - 2\mathrm{g}z_0x_0}{(ca)^2} = \frac{\mathrm{a}x_0^2 + \mathrm{b}y_0^2 - 2\mathrm{h}x_0y_0}{(ab)^2}$$
$$= \frac{\mathrm{a}x_0^2 + \mathrm{d}w_0^2 - 2\mathrm{l}x_0w_0}{(ad)^2} = \frac{\mathrm{b}y_0^2 + \mathrm{d}w_0^2 - 2\mathrm{m}y_0w_0}{(bd)^2}$$
$$= \frac{\mathrm{c}z_0^2 + \mathrm{d}w_0^2 - 2\mathrm{n}z_0w_0}{(cd)^2}.$$

231. It was shown (Art. 214) that by forming the condition that $\alpha x + \beta y + \gamma z + \delta w$ should touch $U + \lambda V$, we get an equation in λ whose coefficients are the invariants *in plano* Δ, Δ', Θ, Θ' of the sections of U and V by the given plane. It was also shown (*Conics*, Art. 382) that if we form the invariants of any conic and the pair of circular points at infinity, $\Theta = 0$ is the condition that the curve should be a parabola, $\Theta' = 0$ the condition that it should be an equilateral hyperbola, and $\Theta'^2 = 4\Theta\Delta'$ the condition that the curve should pass

through either circular point at infinity. Applying then these principles to any quadric in rectangular coordinates and the tangential equation of the imaginary circle $\alpha^2+\beta^2+\gamma^2$, we get for the *condition*, $\Theta=0$, *that any section should be a parabola*,

$$(bc-f^2)\alpha^2+(ca-g^2)\beta^2+(ab-h^2)\gamma^2$$
$$+2(gh-af)\beta\gamma+2(hf-bg)\gamma\alpha+2(fg-ch)\alpha\beta=0;$$

for the *condition* $\Theta'=0$ *that it should represent an equilateral hyperbola*

$$(b+c)\alpha^2+(c+a)\beta^2+(a+b)\gamma^2-2f\beta\gamma-2g\gamma\alpha-2h\alpha\beta=0,$$

while $\Theta'^2=4\Theta\,(\alpha^2+\beta^2+\gamma^2)$ is the *condition that the plane should pass through any of the four points at infinity common to the quadric and any sphere.*

232. We know from the theory of conics that if $\sigma=0$ be the tangential equation of a conic, and $\sigma'=0$ the tangential equation of the two circular points at infinity in its plane, $\sigma+\lambda\sigma'=0$ is the tangential equation of any confocal conic. Now the tangential equation of the pair of points where the imaginary circle $\alpha^2+\beta^2+\gamma^2$ is met by the plane $\alpha'x+\beta'y+\gamma'z+\delta'w$ is evidently

$$(\alpha'^2+\beta'^2+\gamma'^2)\ (\alpha^2+\beta^2+\gamma^2)-(\alpha\alpha'+\beta\beta'+\gamma\gamma')^2=0.$$

Thus then the *tangential equation of all conics confocal to the section by* $\alpha'x+\beta'y+\gamma'z+\delta'w$ *of* $ax^2+by^2+cz^2+dw^2$, is

$$\alpha^2\,\{(cd\beta'^2+db\gamma'^2+bc\delta'^2)\ +\lambda(\beta'^2+\gamma'^2)\}$$
$$+\beta^2\,\{(cd\alpha'^2+da\gamma'^2+ac\delta'^2)+\lambda(\alpha'^2+\gamma'^2)\}$$
$$+\gamma^2\,\{(bd\alpha'^2+da\beta'^2+ab\delta'^2)+\lambda(\alpha'^2+\beta'^2)\}$$
$$+\delta^2(bc\alpha'^2+ca\beta'^2+ab\gamma'^2)-2(ad+\lambda)\beta'\gamma'\beta\gamma$$
$$-2(bd+\lambda)\ \gamma'\alpha'\gamma\alpha-2(cd+\lambda)\alpha'\beta'\alpha\beta$$
$$-2bc\alpha'\delta'\alpha\delta-2ca\beta'\delta'\beta\delta-2ab\gamma'\delta'\gamma\delta=0.$$

If we form the reciprocal of this according to the ordinary rules, we get the square of $\alpha'x+\beta'y+\gamma'z+\delta'w$ multiplied by $\Sigma^2+\lambda\Sigma\Theta'+\lambda^2\ (\alpha'^2+\beta'^2+\gamma'^2)\ \Theta$ where Σ is the condition that $\alpha'x+\beta'y+\gamma'z+\delta'w$ should touch the given quadric, and Θ', Θ have the same signification as in the last article. By equating the second factor to nothing we obtain the values of λ which give the *tangential equations of the foci of the plane section* in question.

Ex. 1. To find the foci of the section of $4x^2 + y^2 - 4z^2 + 1$ by $x + y + z$. The equation for λ is found to be $3\lambda^2 + 2\lambda = 16$, whence $\lambda = 2$ or $= -\frac{8}{3}$. The equation of the last article, for the values $\alpha' = \beta' = \gamma' = 1$, and the given values of a, b, c, d, is

$$\alpha^2(-3 + 2\lambda) + 2\lambda\beta^2 + (5 + 2\lambda)\gamma^2 - 16\delta^2 - 2(4+\lambda)\beta\gamma - 2(1+\lambda)\gamma\alpha + 2(4-\lambda)\alpha\beta = 0.$$

Substituting $\lambda = 2$ it becomes $(\alpha + 2\beta - 3\gamma)^2 - 16\delta^2$, whence the coordinates of the foci are $\pm\frac{1}{4}$, $\pm\frac{2}{4}$, $\mp\frac{3}{4}$. The other value of λ gives the imaginary foci.

Ex. 2. To find the locus of the foci of all central sections of the quadric $ax^2 + by^2 + cz^2 + 1$. Making $\delta' = 0$, the equation for λ is found to be

$$\frac{\alpha'^2}{a + \lambda} + \frac{\beta'^2}{b + \lambda} + \frac{\gamma'^2}{c + \lambda} = 0.$$

By the help of this relation the tangential equation of the foci is reduced to the form

$$\left(\frac{\alpha\alpha'}{a + \lambda} + \frac{\beta\beta'}{b + \lambda} + \frac{\gamma\gamma'}{c + \lambda}\right)^2 - \frac{bc\alpha'^2 + ca\beta'^2 + ab\gamma'^2}{(a + \lambda)(b + \lambda)(c + \lambda)}\delta^2 = 0.$$

Thus then the coordinates of the foci are

$$x = \frac{\alpha'}{a + \lambda},\ y = \frac{\beta'}{b + \lambda},\ z = \frac{\gamma'}{c + \lambda},\ w^2 = \frac{bc\alpha'^2 + ca\beta'^2 + ab\gamma'^2}{(a + \lambda)(b + \lambda)(c + \lambda)}.$$

Solving for α', β', γ' from the first three equations and substituting in the equation for λ, we get

$$(ax^2 + by^2 + cz^2) + \lambda(x^2 + y^2 + z^2) = 0;$$

solving for λ and substituting in the value for w^2, we get the equation of the locus, viz.

$$(x^2 + y^2 + z^2)\,[bcx^2\{(a - b)y^2 + (a - c)z^2\}^2 + cay^2\{(b - c)z^2 + (b - a)x^2\}^2 + abz^2\{(c - a)x^2 + (c - b)y^2\}^2]$$
$$= w^2\{(a - b)y^2 + (a - c)z^2\}\{(b - c)z^2 + (b - a)x^2\}\{(c - a)x^2 + (c - b)y^2\},$$

a surface of the eighth degree having the centre of the given quadric as a multiple point.

The left-hand side of the equation may be written in the simpler form

$$(x^2 + y^2 + z^2)\,(ax^2 + by^2 + cz^2)\,\{a(b - c)^2y^2z^2 + b(c - a)^2z^2x^2 + c(a - b)^2x^2y^2\}.$$

For a discussion of this surface see a paper by Painvin, *Nouvelles Annales*, Second Series, III. p. 481.

From the property that if a point be a focus of a plane section of a quadric, the plane is a cyclic plane of the tangent cone from the point, M'Cay writes down immediately this locus in the coordinate system of Art. 160.

In fact the equation of the tangent cone (171) being

$$\frac{x^2}{a^2 - a'^2} + \frac{y^2}{a^2 - a''^2} + \frac{z^2}{a^2 - a'''^2} = 0,$$

one of its pairs of cyclic planes is

$$\frac{a'^2 - a''^2}{a^2 - a'^2}\,x^2 = \frac{a''^2 - a'''^2}{a^2 - a'''^2}\,z^2.$$

But, for central sections, since the coordinates of the centre satisfy this equation, we may replace x by p' and z by p''', Art. 165. Substituting these values, we get

$$\frac{a'^2b'^2c'^2}{a^2 - a'^2} = \frac{a'''^2b'''^2c'''^2}{a^2 - a'''^2} \qquad\qquad (1).$$

It is easily derived from this by the cubic equation of Art. 158, taking $a^2 - a'^2 = \lambda^2$, $a^2 - a'''^2 = \nu^2$, and μ^2 the third root, that $\mu^2 = -\dfrac{\rho^2 S}{S+1}$, where $\rho^2 = x^2 + y^2 + z^2$, and $S = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$; and this value of μ^2 substituted in the cubic gives an equation of the eighth degree in x, y, z as above.

Ex. 3. To find the locus of foci of sections parallel to an axis (say $\alpha' = 0$). The equation which must break up into factors is in this case

$$\alpha^2\{(c+\lambda)\beta'^2 + (b+\lambda)\gamma'^2 + bc\delta'^2\} + \beta^2\{(a+\lambda)\gamma'^2 + ac\delta'^2\} + \gamma^2\{(a+\lambda)\beta'^2 + ab\delta'^2\}$$
$$+ \delta^2 a(c\beta'^2 + b\gamma'^2) - 2(a+\lambda)\beta'\gamma'\beta\gamma - 2ca\beta'\delta'\beta\delta - 2ab\gamma'\delta'\gamma\delta = 0.$$

The condition that the resolution into factors shall be possible is

$$(a+\lambda)(b\gamma'^2 + c\beta'^2) + abc\delta'^2 = 0.$$

Subject to this condition the equation becomes

$$\frac{\alpha^2}{bc(a+\lambda)}\{(c+\lambda)\beta'^2 + (b+\lambda)\gamma'^2 + bc\delta'^2\} = \left\{\frac{\beta\beta'}{b} + \frac{\gamma\gamma'}{c} + \frac{\alpha\delta\delta'}{a+\lambda}\right\}^2,$$

whence $\beta' = by$, $\gamma' = cz$, $a\delta' = (a+\lambda)w$, substituting which values in the equation of condition we have $(a+\lambda)w^2 + acz^2 + aby^2 = 0$; whence again substituting in

$$bc(a+\lambda)x^2 = (c+\lambda)\beta'^2 + (b+\lambda)\gamma'^2 + bc\delta'^2,$$

we get for the required locus

$$(by^2 + cz^2)\{b^2(a-c)y^2 + c^2(a-b)z^2 - abcx^2\} + w^2\{b^2(a-c)y^2 + c^2(a-b)z^2\} = 0.$$

It is obvious that the methods of this and the preceding article can be applied to equations in quadriplanar coordinates.

233. *Given four quadrics the locus of a point whose polar planes with respect to all four meet in a point is a surface of the fourth degree, which we call the Jacobian of the system of quadrics* (see *Conics*, Art. 388). Its equation in fact is evidently got by equating to nothing the determinant formed with the four sets of differential coefficients U_1, U_2, U_3, U_4; V_1, V_2, &c. It is evident that when the polars of any point with regard to U, V, W, T meet in a point, the polar with respect to $\lambda U + \mu V + \nu W + \pi T$ will pass through the same point. [The Jacobian may also be defined as the locus of the foci of the involutions on lines which are cut in involution by the four quadrics U, V, W, T. For if A is a point on the Jacobian and B the intersection of its four polar planes, the line AB is cut harmonically by all the quadrics. Obviously B is also on the Jacobian, and A and B are called *corresponding* points.]

The Jacobian is also the locus of the vertices of all cones

which can be represented by $\lambda U+\mu V+\nu W+\pi T$. Thus, then, *given six points the locus of the vertices of all cones of the second degree which can pass through them is a surface of the fourth degree.* For if T, U, V, W be any quadrics through the six points, every quadric through them can be represented by $\lambda U+\mu V+\nu U+\pi T$, since this last form contains the three independent constants which are necessary to complete the determination of the surface. It is geometrically obvious that *this quartic surface passes through each of the fifteen lines joining any two of the given points, and also through each of the ten lines which are the intersections of two planes passing through the given points.*

[The six points are nodes on the quartic. More generally, if U, V, W, T have a common point, it is a node on the Jacobian ($J=0$). For if we express the condition that the common point (A) is 1, 0, 0, 0, it is easy to see, without expanding the determinant, that the coordinates satisfy

$$J=0,\ \frac{dJ}{dx}=0,\ \frac{dJ}{dy}=0,\ \&c.$$

And therefore (Art. 283) A is a node. In fact A coincides with its correspondent, and therefore every line through A meets J in two coincident points.]*

If in any case $\lambda U+\mu V+\nu W+\pi T$ can represent two planes, the intersection of those planes lies on the Jacobian.

If the four surfaces have a common self-conjugate tetrahedron the Jacobian reduces to four planes. For let the surfaces be $ax^2+by^2+cz^2+dw^2$, $a'x^2+b'y^2+$ &c., &c., then we have $U_1=ax$, $V_1=a'x$, &c., and it is easy to see that the Jacobian is $xyzw$ multiplied by the determinant $(ab'c''d''')$. [The converse is also true.]

If one of the quantities U be a perfect square L^2, L is a factor in U_1, U_2, &c., and the Jacobian consists of a plane and a surface of the third order. If the surfaces have common four points in a plane, it is evident geometrically that this plane is part of the Jacobian; and if they have a plane section

[* The locus of the vertices of cones through six points is known as "Weddle's Quartic"; it was first discussed by Weddle, *Cambridge and Dublin M.J.*, 5 (1850), and subsequently by Cayley, *Comptes Rendus*, 52 (1861), by Hutchinson, *Annals of Maths.*, II. (1897), and by Bateman, *Proc. Lond. Math. Soc.*, II. 3 (1905).]

common to all, this plane counts doubly in the Jacobian, which is only a surface of the second degree besides. Thus *the Jacobian of four spheres is a sphere cutting the others at right angles.* [If A be a point on the Jacobian sphere S, the corresponding point is at the opposite extremity of the diameter through A.]

Cor. If a surface of the system $\lambda U+\mu V+\nu W$ touch T, the point of contact is evidently a point on the locus considered in this article, and therefore lies somewhere on the curve of intersection of T with the Jacobian. Again, if a surface of the system $\lambda U+\mu V$ touch the curve of intersection of T, W; that is to say, if at one of the points where $\lambda U+\mu V$ meets T, W, the tangent plane to the first pass through the intersection of the tangent planes to the two others, the point of contact is evidently a point on the Jacobian of the system. It follows that *sixteen surfaces of the system* $\lambda U+\mu V$ *can be drawn to touch the curve* T, W; for since three surfaces of degrees m, n, p meet in mnp points, the Jacobian, which is of the fourth degree, meets the intersection of the two quadrics T, W in sixteen points.

234. *To reduce a pair of quadrics* U, V *to the canonical form* $x^2+y^2+z^2+w^2$, $ax^2+by^2+cz^2+dw^2$. In the first place the constants a, b, c, d are given by the biquadratic

$$\Delta\lambda^4-\Theta\lambda^3+\Phi\lambda^2-\Theta'\lambda+\Delta'=0.$$

Then solving the equations

$$x^2+y^2+z^2+w^2=U,\ a\ (bc+cd+db)\ x^2+\&c.=T,$$
$$a\ (b+c+d)\ x^2+\&c.=T',\ ax^2+\&c. \qquad =V,$$

we find x^2, y^2, z^2, w^2, in terms of the known functions U, V, T, T'. Strictly speaking we ought to commence by dividing U and V by the fourth root of Δ, in order to reduce them to a form in which the discriminant of U shall be 1. But it will come to the same thing if leaving U and V unchanged we divide by Δ, T and T' as calculated from the coefficients of the given equation.

Ex. 1. To reduce to the canonical form

$$5x^2-11y^2-11z^2-6w^2+24yz+22zx-20xy+8yw+4zw=0,$$
$$25x^2-10y^2-15z^2-5w^2+88yz+46zx-30xy-10xw+10yw+18zw=0.$$

The reciprocals of these equations are

$$550\alpha^2 + 1036\beta^2 + 850\gamma^2 - 324\delta^2 + 2120\beta\gamma + 500\gamma\alpha - 520\alpha\beta - 180\alpha\delta + 2088\beta\delta + 1980\gamma\delta = 0,$$

$$3950\alpha^2 + 800\beta^2 + 2750\gamma^2 - 9720\delta^2 + 11200\beta\gamma + 4900\gamma\alpha - 4160\alpha\beta + 25920\beta\delta + 16200\gamma\delta = 0.$$

And the biquadratic is

$$8100\,\{\lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24\} = 0;$$

whence a, b, c, d are 1, 2, 3, 4. We then calculate T and T' by the formula

$$T = x^2\{B'(ab - h^2) + C'(ac - g^2) + D'(ad - l^2) + 2F'(af - gh) + 2M'(am - hl) + 2N'(an - gl)\}$$
$$+ 2yz\{A'(af - gh) + D'(df - mn) + M'(mf - bn) + N'(nf - cm) + G'(fg - ch) + H'(fh - bg) + F'(f^2 - bc) + L'(2lf - mg - nh)\} + \&c.,$$

and dividing T and T' so calculated by Δ ($= 8100$), we write

$$X^2 + Y^2 + Z^2 + W^2 = 5x^2 - 11y^2 - 11z^2 - 6w^2 + 24yz + 22zx - 20xy + 8yw + 4zw,$$

$$X^2 + 2Y^2 + 3Z^2 + 4W^2 = 25x^2 - 10y^2 - 15z^2 - 5w^2 + 38yz^2 + 46zx - 30xy - 10xw + 10yw + 18zw,$$

$$9X^2 + 16Y^2 + 21Z^2 + 24W^2 = 161x^2 - 100y^2 - 135z^2 - 55w^2 + 306yz + 342zx - 250xy - 70xw + 70yw + 126zw,$$

$$26X^2 + 38Y^2 + 42Z^2 + 44W^2 = 280x^2 - 300y^2 - 360z^2 - 170w^2 + 772yz + 776zx - 628xy - 108xw + 180yw + 252zw.$$

Then from $24U - V + T' - T$, we get

$$6X^2 = -6\{2x + 3y - 2z - 2w\}^2.$$

And, in like manner,

$$Y^2 = -(x + 2y - 3z + 2w)^2,\ Z^2 = (3x - y + z - w)^2,\ W^2 = (x + y + z + w)^2.$$

Ex. 2. It having been shown that x^2, y^2, z^2, w^2 can be expressed in terms of U, V, T, T', it follows that the square of the Jacobian of these four surfaces can also be expressed as a function of them. We find thus

$$J^2 = \Delta T^4 - \Theta T^3T' + \Phi T^2T'^2 - \Theta' TT'^3 + \Delta' T'^4$$
$$+ V\{(\Theta^2 - 2\Delta\Phi)\,T^3 + (\Theta\Phi - 3\Theta'\Delta)\,T^2T' + (\Theta\Theta' - 4\Delta\Delta')\,TT'^2 - \Delta'\Theta T'^3\}$$
$$+ U\{(\Theta'^2 - 2\Delta'\Phi)\,T'^3 + (\Theta'\Phi - 3\Theta\Delta')\,T'^2T + (\Theta\Theta' - 4\Delta\Delta')T'T^2 - \Delta\Theta' T^3\}$$
$$+ \Delta V^2\{(\Phi^2 - 2\Theta\Theta' + 2\Delta\Delta')T^2 - (\Theta'\Phi - 3\Theta\Delta')TT' + \Phi\Delta' T'^2\}$$
$$+ UV\{(2\Delta\Phi\Theta' + \Delta\Delta'\Theta - \Theta^2\Theta')T^2 + [(\Theta\Theta' + 4\Delta\Delta')\Phi - 3(\Delta\Theta'^2 + \Delta\,\Theta^2)]TT' + (2\Delta'\Phi\Theta' + \Delta\Delta'\Theta' - \Theta\Theta'^2)T'^2\}$$
$$+ \Delta' U^2\{(\Phi^2 - 2\Theta\Theta' + 2\Delta\Delta')\,T'^2 - (\Theta\Phi - 3\Delta\Theta')\,TT' + \Delta\Phi T^2\}$$
$$+ T\{(\Theta'^2 - 2\Delta'\Phi)\,V^3\Delta^2 - (\Theta'\Phi^2 - 2\Theta\Theta'^2 + 5\Theta'\Delta'\Delta - \Theta\Phi\Delta')\,V^2U\Delta + (\Theta^2\Phi - 2\Phi^2\Delta - \Theta\Theta'\Delta + 4\Delta'\Delta^2)\Delta' VU^2 - \Delta\Delta'^2\Theta U^3\}$$
$$+ T'\{(\Theta^2 - 2\Delta\Phi)\,U^3\Delta'^2 - (\Theta\Phi^2 - 2\Theta'\Theta^2 + 5\Theta\Delta\Delta' - \Theta'\Phi\Delta)\,U^2V\Delta' + (\Theta'^2\Phi - 2\Phi^2\Delta' - \Theta\Theta'\Delta' + 4\Delta\Delta'^2)\,\Delta UV^2 - \Delta^2\Delta'\Theta' V^3\}$$
$$+ \Delta^3\Delta'^2V^4 + \Delta^2\Delta'^3U^4 - UV^3\Delta^2\{\Theta'^3 - 3\Theta'\Phi\Delta' + 3\Theta\Delta'^2\} - U^3V\Delta'^2\{\Theta^3 - 3\Theta\Phi\Delta + 3\Theta'\Delta^2\} + \Delta\Delta' U^2V^2\{\Phi^3 - 3\Phi\Delta\Delta' + 3\Theta^2\Delta' + 3\Theta'^2\Delta - 3\Theta\Theta'\Phi\}.$$

Ex. 3. The formulæ for the coordinates of a point on the curve UV, given Art. 220*d*, evidently result from the determination of this article. We proceed to treat similarly the tangential equations.

Writing down the four contravariants (214) in the form

$$\begin{aligned}
&\alpha^2.\, b'c'd' + \beta^2.\, c'd'a' + &&= \sigma',\\
&\alpha^2.\, (bc'd' + cd'b' + db'c') + \beta^2\,(\quad) + &&= \tau',\\
&\alpha^2.\, (cdb' + dbc' + bcd') + \beta^2\,(\quad) + &&= \tau,\\
&\alpha^2.\, bcd + \beta^2.\, cda + &&= \sigma,
\end{aligned}$$

these give, when solved for α^2, β^2, γ^2, δ^2,

$$(ab')\,(ac')\,(ad')\,\alpha^2 = a^3\sigma' - a^2a'\tau' + aa'^2\tau - a'^3\sigma, \text{ \&c.}$$

Hence, for any tangent plane common to U and V,

$$(ab')\,(ac')\,(ad')\,\alpha^2 = aa'\,(a'\tau - a\tau'), \text{ \&c.}$$

The coordinates of the line in which this intersects a consecutive common tangent plane, *i.e.* the coordinates of a generator of the circumscribed developable, are derived from these by taking the consecutive tangent plane

$$2\frac{d\alpha}{\alpha} = \frac{a'd\tau - ad\tau'}{a'\tau - a\tau'},\quad 2\frac{d\beta}{\beta} = \frac{b'd\tau - bd\tau'}{b'\tau - b\tau'}, \text{ \&c.},$$

whence, by taking the difference of these two and substituting for α, β, we get the value for the coordinate

$$\rho^2 = aa'bb'\,(ab')\,(cd')\,(c'\tau - c\tau')\,(d\,\tau - d\tau'),$$

and for the other coordinates values corresponding, omitting a common factor. From these the tangential equation of the circumscribed developable may be found.

Systems of three Quadrics.

235. If we form the discriminant of $\lambda U + \mu V + \nu W$, the coefficients of the several powers of λ, μ, ν will evidently be invariants of the system U, V, W. There are three invariants however of this system (which we shall call Δ*, I, J)

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* In the former editions it had been supposed that the equations of any three quadrics could be reduced to the form

$$\begin{aligned}
U &= a\,x^2 + b\,y^2 + c\,z^2 + d\,u^2 + e\,v^2,\\
V &= a'\,x^2 + b'\,y^2 + c'\,z^2 + d'\,u^2 + e'\,v^2,\\
W &= a''x^2 + b''y^2 + c''z^2 + d''u^2 + e''v^2,
\end{aligned}$$

a form containing 12 independent constants expressed and 15 implicitly, or, in all, the right number 27 (see Art. 141). Doubt was cast on the validity of this argument when Clebsch observed that a similar argument does not hold good for plane quartics. The form

$$ax^4 + by^4 + cz^4 + du^4 + ev^4,$$

contains the right number of constants for representing a general quartic; yet for this form it is easily shown that an invariant vanishes which in general is not $= 0$ (see *Higher Plane Curves*, Art. 294). The same thing is true of the form

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{u} + \frac{e}{v},$$

which though containing the right number of constants will not represent a quartic in general, but only one for which a certain invariant relation is fulfilled. Frahm showed (*Math. Annal*, VII.) that there is in fact an intimate relation between the theory of three quadrics and that of a plane quartic.

which deserve special attention as being also invariants of any three quadrics of the system $\lambda U + \mu V + \nu W$; or, what is the same thing, as being also *combinants*.

The *invariant A vanishes when each of the three quadrics U, V, W is the polar quadric of a point with regard to the same surface of the third degree.* In fact it is easy to see that, taking two points, 1, 2 and a cubic surface, the polar plane of 1 with respect to the polar quadric of 2 must be the same as the polar plane of 2 with regard to the polar quadric of 1. Supposing then U, V, W to be the polar quadrics of points 1, 2, 3 respectively, and expressing that the polar plane of 1 in respect of V is identical with that of 2 in respect of U, we get by comparing coefficients of x, y, z, w four equations linear in x_1, y_1, x_2, &c. Similarly two other sets of four are got by comparing the surfaces U, W; V, W. Eliminating then linearly the twelve unknown variables x_1, y_1, &c., x_2, &c., the result of elimination can be written

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Form the discriminant of $\lambda U + \mu V + \nu W$ and we get a result which is a ternary quartic in λ, μ, ν of the most general kind. Now the discriminant of

$$ax^2 + by^2 + cz^2 + du^2 + ev^2,$$

where we suppose that $x + y + z + u + v \equiv 0$, is easily seen to be

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 0.$$

And, therefore, if U, V, W be three quadrics of this form the discriminant of $\lambda U + \mu V + \nu W$ is got by writing $\lambda a + \mu a' + \nu a''$ for a &c., in the above. And according to what has been just stated this is only a ternary quartic of a special form. If then we write down the invariant condition that the discriminant of $\lambda U + \mu V + \nu W$ considered as a ternary quartic in λ, μ, ν should be capable of being reduced to the special form just mentioned, we have at the same time the condition that these quadrics should be such that their equations may be written as the sum of squares of the same five linear functions. Toeplitz (*Math. Annal*, XI.) gave the form of A definitely as in the text, and also by determining its symbolical expression showed that it can be expressed in terms of the functions of the coefficients which occur in the conditions that a right line should touch U, V, W respectively. The condition that a line should touch a surface may be expressed symbolically (see Arts. 80, 217) as $(12\alpha\beta)^2$. The symbolical function $(12\alpha\beta)(12\alpha'\beta')$ expresses that two lines are harmonic conjugates with regard to a surface, and is a function of the same coefficients of the quadric. And, if taking α, β; α', β' as symbols with respect to two other surfaces we multiply by $(\alpha\beta\alpha'\beta')$ we get the symbol which expresses A.

at once as a determinant, but as this is a skew symmetrical determinant of even order, it is a perfect square; thus the condition in question is of the second order only, in the coefficients of each of the surfaces. Reducing this determinant by assuming two of the surfaces in the forms

$$a'x^2 + b'y^2 + c'z^2 + d'w^2,$$
$$a''x^2 + b''y^2 + c''z^2 + d''w^2,$$

which is always admissible; it is found to be in this case

$$\begin{vmatrix} 0, & (b'a'')h, & (c'a'')g, & (d'a'')l \\ (a'b'')h, & 0, & (c'b'')f, & (d'b'')m \\ (a'c'')g, & (b'c'')f, & 0, & (d'c'')n \\ (a'd'')l, & (b'd'')m, & (c'd'')n, & 0 \end{vmatrix},$$

which is also skew symmetrical and is the square of

$$(b'c'')\ (a'd'')\ fl + (c'a'')\ (b'd'')\ gm + (a'b'')\ (c'd'')\ hn.$$

In this form it is easily seen that A *vanishes if* U, V, W *each admit of being written as sum of five squares.* In fact we can in this case eliminate one variable between each pair of equations reducing two to the forms just written, making each of them the sum of four squares; and the third becomes, by replacing the fifth variable from the universal linear relation,

$$ax^2 + by^2 + cz^2 + dw^2 + e\ (x + y + z + w)^2 = 0$$

whence $fl = gm = hn = e^2$, and these values substituted in the expression just found for A evidently make it vanish.

236. The invariant which we call I *vanishes, whenever any four of the points of intersection of* U, V, W *lie in a plane* (a condition which implies that the other four points of intersection lie in a plane), or, in other words, whenever it is possible to find values of λ, μ, ν, which will make $\lambda U + \mu V + \nu W$ represent two planes. Now in this case the tangential equation vanishes (Art. 214), hence, writing for a, $\lambda a + \mu a' + \nu a''$, &c. in σ, let the result be denoted by

$$\sigma_{000}\lambda^3 + \sigma_{001}\lambda^2\mu + \sigma_{002}\lambda^2\nu + \ .\ .\ .\ = 0,$$

the ten coefficients of this quadric in α, β, γ, δ, therefore vanish, whence we can write down the required condition as the determinant of the tenth order got by eliminating λ, μ, ν;

but each coefficient is of the third order in the original coefficients, hence this invariant, involving symmetrically each surface, must be of the tenth degree in the coefficients of each surface (compare *Conics*, 389*a*). That I is of the tenth degree in the coefficients of each surface may be otherwise seen as follows: Let U, U', V, W be four quadrics passing each through the same six points; then since through these points twenty planes [ten pairs of planes] can be drawn, it follows that the problem to determine λ, μ, ν so that $U+\lambda U'+\mu V+\nu W$ may represent two planes, admits of ten solutions. But λ might also be determined by forming the invariant I of the system U, V, W, and then substituting for each coefficient a of U, $a+\lambda a'$. And since there are ten values of λ, the result of substitution must contain λ in the tenth degree; and therefore I must contain the coefficients of U in the same degree.

237. The *invariant which we call J vanishes, whenever any two of the eight points of intersection of the surfaces U, V, W coincide.** Thus, if at any point common to the three surfaces, their three tangent planes pass through a common line, the consecutive point on this line will also be common to all the surfaces. Such a point will also be the vertex of a cone of the system $\lambda U+\mu V+\nu W$. For take the point as origin, and if the tangent planes be x, y, $ax+by$, the equations of the surfaces are $x+u_2$, $y+v_2$, $ax+by+w_2$, where u_2, v_2, w_2 denote terms of the second degree. And it is evident that $aU+bV-W$ is a cone having the origin for its vertex.

The invariant J is of the sixteenth degree in the coefficients of each of the surfaces. For if in J we substitute for each coefficient a of U, $a+\lambda a'$ where a' is the corresponding coefficient of another surface U', it is evident that the degree of the result in λ is the same as the number of surfaces of the

* This invariant is called by Cayley the tact-invariant of a system of three quadrics, as that considered Art. 202 is the tact-invariant of a system of two.

system $U+\lambda U'$ which can be drawn to touch the curve of intersection of V, W; that is to say, sixteen (Cor., Art. 233).

238. If $ax^2+by^2+cz^2+du^2+ev^2$ represent a cone, the coordinates of the vertex satisfy the four equations got by differentiating with respect to x, y, z, u; that is to say, (remembering that $x+y+z+u+v$ is supposed to $=0$) $ax=ev$, $by=ev$, &c. The coordinates of the vertex may then be written $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$, substituting which values in the condition connecting x, y, z, u, v, we obtain the discriminant of the surface, viz.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=0.$$

Thus, then, when the equations of U, V, W admit of being written in the form here used, the discriminant of

$$\lambda U+\mu V+\nu W \text{ is}$$

$$\frac{1}{\lambda a+\mu a'+\nu a''}+\frac{1}{\lambda b+\mu b'+\nu b''}+\text{\&c.}=0;$$

and when $\lambda U+\mu V+\nu W$ represents a cone, if we substitute the coordinates of its vertex in the equation of each of the surfaces in succession, we get

$$\frac{a}{(\lambda a+\mu a'+\nu a'')^2}+\frac{b}{(\lambda b+\mu b'+\nu b'')^2}+\text{\&c.}=0,$$

$$\frac{a'}{(\lambda a+\mu a'+\nu a'')^2}+\frac{b'}{(\lambda b+\mu b'+\nu b'')^2}+\text{\&c.}=0, \text{ \&c.}$$

But these equations are the differentials of the discriminant with respect to λ, μ, ν. Hence we derive the theorem that in the case in question if we form the discriminant of

$$\lambda U+\mu V+\nu W,$$

and then the discriminant of this again with respect to λ, μ, ν; J will be a factor in the result. It may be shown easily that I must also be a factor in this result, and the result is in fact I^2J.

238*a*. *Given three quadrics the locus of a point whose polar planes with respect to all three meet in a line is a curve of the sixth order*, which may be called *the Jacobian curve* of

the system. For such a point must evidently satisfy all the equations got by equating to nothing the determinants of the system of differential coefficients U_1 &c., of U, V_1 &c., of V, &c.,

$$\left\| \begin{matrix} U_1, & U_2, & U_3, & U_4 \\ V_1, & V_2, & V_3, & V_4 \\ W_1, & W_2, & W_3, & W_4 \end{matrix} \right\|,$$

but equating to zero any two of these determinants as (123) and (124) we get two surfaces of the third order which have common the cubic curve (Art. 134) whose equations are got by the vanishing of

$$\left\| \begin{matrix} U_1, & V_1, & W_1 \\ U_2, & V_2, & W_2 \end{matrix} \right\|$$

and this does not belong to the other cubic surfaces. Hence there is only a sextic curve common.

[The Jacobian curve is evidently the locus of the vertices of cones passing through the eight points of intersection of the three given quadrics.]

238*b*. If we express the *condition that the right line joining the points* 1 *and* 2 *may be cut in involution by three quadrics* U, V, W, writing the quadratic of Art. 75 in the form

$$U_{11}\lambda^2 + 2U_{12}\lambda\mu + U_{22}\mu^2 = 0, \text{ \&c.}$$

that condition is found by expressing that there are two points on the line such that the polar plane of each, with respect to all three quadrics, passes through the other: hence

$$M = \begin{vmatrix} U_{11}, & U_{12}, & U_{22} \\ V_{11}, & V_{12}, & V_{22} \\ W_{11}, & W_{12}, & W_{22} \end{vmatrix} = 0,$$

but this may be written in the form

$$0 = \left\| \begin{matrix} a, & b, & c, & d, & f \dots \\ a', & b', & c', & d', & f' \dots \\ a'', & b'', & c'', & d'', & f'' \dots \end{matrix} \right\| \left\| \begin{matrix} x_1^2, & y_1^2, & z_1^2, & w_1^2, & 2y_1z_1 \dots \\ x_1x_2, & y_1y_2, & z_1z_2, & w_1w_2, & (y_1z_2 + y_2z_1) \dots \\ x_2^2, & y_2^2, & z_2^2, & w_2^2, & 2y_2z_2 \dots \end{matrix} \right\|,$$

and it can be seen without difficulty that each determinant in the second matrix consists of powers and products of the six coordinates of the right line 1, 2. Hence we have the relation

in question as *a complex of the third order* the coefficients of which are linear in the coefficients of each quadric. Employing a usual method of squaring, we find by multiplying

$$\begin{vmatrix} U_{11}, & U_{12}, & U_{22} \\ V_{11}, & V_{12}, & V_{22} \\ W_{11}, & W_{12}, & W_{22} \end{vmatrix} \begin{vmatrix} U_{22}, & -2U_{12}, & U_{11} \\ V_{22}, & -2V_{12}, & V_{11} \\ W_{22}, & -2W_{12}, & W_{11} \end{vmatrix} = \begin{vmatrix} 2\Psi_{00}, & \Psi_{10}, & \Psi_{20} \\ \Psi_{01}, & 2\Psi_{11}, & \Psi_{21} \\ \Psi_{02}, & \Psi_{12}, & 2\Psi_{22} \end{vmatrix}$$

where Ψ_{00} is the condition for the line to touch U, &c. and Ψ_{01} for it to be cut harmonically by U and V, &c. (Art. 217). Hence it is seen that the squares and products of the coefficients in M can be expressed by the combinations of the original coefficients which arise from the second minors of the discriminant Ex. 6, Art. 200. Again, the complex M is the same for any three surfaces of the system $\lambda U + \mu V + \nu W$. Also $M = 0$ if for such a surface we have $\lambda U_{11} + \mu V_{11} + \nu W_{11} = 0$, $\lambda U_{12} + \mu V_{12} + \nu W_{12} = 0$, $\lambda U_{22} + \mu V_{22} + \nu W_{22} = 0$, hence (Art. 80*g*) it contains all the right lines which are contained in surfaces of the system. This complex M may be also written in axial coordinates: Toeplitz has noticed that when the products of corresponding coefficients of both forms are summed, the invariant A is the result.

[Ex. Determine the number of lines that may be drawn through a given point so as to be cut in involution by four quadrics.]

CHAPTER X.

CONES AND SPHERO-CONICS.

239. If a cone of any degree be cut by any sphere, whose centre is the vertex of the cone, the curve of section will evidently be such that the angle between two edges of the cone is measured by the arc joining the two corresponding points on the sphere. When the cone is of the second degree, the curve of section is called a *sphero-conic*. By stating many of the properties of cones of the second degree as properties of sphero-conics, the analogy between them and corresponding properties of conics becomes more striking.*

Strictly speaking, the intersection of a sphere with a cone of the n^{th} degree is a curve of the $2n^{\text{th}}$ degree: but when the cone is concentric with the sphere, the curve of intersection may be divided, in an infinity of ways, into two symmetrical and equal portions, either of which may be regarded as analogous to a plane curve of the n^{th} degree. For if we consider the points of the curve of intersection which lie in any hemisphere, the points diametrically opposite evidently trace out a perfectly symmetrical curve in the opposite hemisphere.†

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* See Chasles's Memoir on Sphero-conics (published in the Sixth Volume of the *Transactions of the Royal Academy at Brussels*, and translated by Graves, Dublin, 1837), from which the enunciations of many of the theorems in this chapter are taken. See also Chasles's later papers *Comptes Rendus*, March and June, 1860.

† It has been remarked (*Higher Plane Curves*, Art. 198) that a cone of any order may comprise two forms of sheet, viz. (1) a twin pair sheet which meets a concentric sphere in a pair of closed curves, such that each point of the one curve is opposite to a point of the other curve (of this kind are cones of the second order); or (2) a single sheet which meets a concentric sphere in a closed curve, such that each point of the curve is opposite to another

Thus, then, a sphero-conic may be regarded as analogous either to an ellipse or to a hyperbola. A cone of the second degree evidently intersects a concentric sphere in two similar closed curves diametrically opposite to each other. One of the principal planes of the cone meets neither curve, and if we look at either of the hemispheres into which this plane divides the sphere, we see a closed curve analogous to an ellipse. The other principal planes divide the sphere into hemispheres containing each hemisphere a half of the two opposite curves, and in particular the principal plane not passing through the focal lines of the cone (suprà, Art. 151) divides the sphere into two hemispheres each containing a curve consisting of two opposite branches like the hyperbola.

The *curve of intersection of any quadric with a concentric sphere is evidently a sphero-conic.*

240. The properties of spherical curves have been studied by means of systems of spherical coordinates formed on the model of Cartesian coordinates. Choose for axes of coordinates any two great circles OX, OY intersecting at right angles, and on them let fall perpendiculars PM, PN from any point P on the sphere. These perpendiculars are not, as in plane coordinates, equal to the opposite sides of the quadrilateral $OMPN$; and therefore it would seem that there is a certain latitude admissible in our selection of spherical coordinates, according as we choose for coordinates the perpendiculars PM, PN, or the intercepts OM, ON which they make on the axes.

Gudermann of Cleves has chosen for coordinates the tangents of the intercepts OM, ON (see Crelle's *Journal*, Vol. VI., p. 240), and the reader will find an elaborate discussion of this system of coordinates in the appendix to Graves's translation of Chasles's Memoir on Sphero-conics. It is easy to see, however, that if we draw a tangent plane to the sphere at the point O, and if the lines joining the centre to the points

point of the curve; (the plane affords an example of such a cone) see Möbius, *Abhandlungen der K. Sächs. Gesellschaft*, Vol. I.

M, N, P meet that plane in points m, n, p; then Om, On will be the Cartesian coordinates of the point p. But Om, On are the tangents of the arcs OM, ON. Hence the equation of a spherical curve in Gudermann's system of coordinates is in reality nothing but the ordinary equation of the plane curve in which the cone joining the spherical curve to the centre of the sphere is met by the tangent plane at the point O.

So, again, if we choose for coordinates the sines of the perpendiculars PM, PN, it is easy to see, in like manner, that the equation of a spherical curve in such coordinates is only the equation of the orthogonal projection of that curve on a plane parallel to the tangent plane at the point O.

It seems, however, to us, that the properties of spherical curves are obtained more simply and directly from the equations of the cones which join them to the centre, than from the equations of any of the plane curves into which they can be projected.

241. Let the coordinates of any point P on the sphere be substituted in the equation of any plane passing through the centre (which we take for origin of coordinates), and meeting the sphere in a great circle AB, the result will be the length of the perpendicular from P on that plane; which varies as the sine of the spherical arc let fall perpendicular from P on the great circle AB. By the help of this principle the equations of cones are interpreted so as to yield properties of spherical curves in a manner precisely corresponding to that used in interpreting the equations of plane curves.

Thus, let α, β be the equations of any two planes through the centre, which may also be regarded as the equations of the great circles in which they meet the sphere, then (as at *Conics*, Art. 54) $\alpha - k\beta$ denotes a great circle, such that the sine of the perpendicular arc from any point of it on α is in a constant ratio to the sine of the perpendicular on β; that is to say, a great circle dividing the angle between α and β into parts whose sines are in the same ratio.

Thus, again, $\alpha - k\beta$, $\alpha - k'\beta$ denote arcs forming with α and β a pencil whose anharmonic ratio is $\frac{k}{k'}$. And $\alpha - k\beta$, $\alpha + k\beta$ denotes arcs forming with α, β a harmonic pencil.

It may be noted here that if A' be the middle point of an arc AB, then B', the fourth harmonic to A', A, and B, is a point distant from A' by 90°. For if we join these points to the centre C, CA' is the internal bisector of the angle ACB, and therefore CB' must be the external bisector. Conversely, if two corresponding points of a harmonic system are distant from each other by 90°, each is equidistant from the other two points of the system.

It is convenient also to mention here that if $x'y'z'$ be the coordinates of any point on the sphere, then $xx' + yy' + zz'$ denotes the great circle having $x'y'z'$ for its pole. It is in fact the equation of the plane perpendicular to the line joining the centre to the point $x'y'z'$.

242. We can now immediately apply to spherical triangles the methods used for plane triangles (*Conics*, Chap. IV., &c.). Thus, if α, β, γ denote the three sides, then $l\alpha = m\beta = n\gamma$ denote three great circles meeting in a point, each of which passes through one of the vertices: while

$$m\beta + n\gamma - l\alpha,\ n\gamma + l\alpha - m\beta,\ l\alpha + m\beta - n\gamma$$

are the sides of the triangle formed by connecting the points where each of these joining lines meets the opposite sides of the given triangle; and $l\alpha + m\beta + n\gamma$ passes through the intersections of corresponding sides of this new triangle and of the given triangle.

The equations $\alpha = \beta = \gamma$ evidently represent the three bisectors of the angles of the triangle. And if A, B, C be the angles of the triangle, it is easily proved that, as in plane triangles, $\alpha \cos A = \beta \cos B = \gamma \cos C$ denote the three perpendiculars. It remains true, as at *Conics*, Art. 54, that if the perpendiculars from the vertices of one triangle on the sides of another meet in a point, so will the perpendiculars from the vertices of the second on the sides of the first.

The three bisectors of sides are

$$\alpha \sin A = \beta \sin B = \gamma \sin C.$$

The arc $\alpha \sin A + \beta \sin B + \gamma \sin C$ passes through the three points where each side is met by the arc joining the middle points of the other two; or, again, it passes through the point on each side 90° distant from its middle point, for $\alpha \sin A \pm \beta \sin B$ meet γ in two points which are harmonic conjugates with the points in which α, β meet it, and since one is the middle point the other must be 90° distant from it (Art. 241). It follows from what has been just said, that the point where $\alpha \sin A + \beta \sin B + \gamma \sin C$ meets any side is the pole of the great circle perpendicular to that side at its middle point, and hence, that the intersection of the three perpendiculars of this kind (that is to say, the centre of the circumscribing circle) is the pole of the great circle

$$\alpha \sin A + \beta \sin B + \gamma \sin C.$$

The equations of the lines joining the vertices of the triangle to the centre of the circumscribing circle are found to be

$$\frac{\alpha}{\sin \frac{1}{2}(B+C-A)} = \frac{\beta}{\sin \frac{1}{2}(C+A-B)} = \frac{\gamma}{\sin \frac{1}{2}(A+B-C)}.$$

243. The condition that two great circles $ax+by+cz$, $a'x+b'y+c'z$ should be perpendicular is manifestly

$$aa'+bb'+cc'=0.$$

The condition that $a\alpha+b\beta+c\gamma$, $a'\alpha+b'\beta+c'\gamma$ should be perpendicular is easily found from this by substituting for α, β, γ their expressions in terms of x, y, z. The result is exactly the same as for the corresponding case in the plane, viz.

$$aa'+bb'+cc'-(bc'+b'c)\cos A-(ca'+c'a)\cos B$$
$$-(ab'+ba')\cos C=0.$$

In like manner the sine of the arc perpendicular to $a\alpha+b\beta+c\gamma$, and passing through a given point is found by substituting the coordinates of that point in $a\alpha+b\beta+c\gamma$ and dividing by the square root of

$$a^2+b^2+c^2-2bc\cos A-2ca\cos B-2ab\cos C.$$

244. Passing now to equations of the second degree, we may consider the equation $\alpha\gamma = m\beta^2$ either as denoting a cone

having α and γ for tangent planes, while β passes through the edges of contact, or as denoting a sphero-conic, having α and γ for tangents, and β for their arc of contact. The equation plainly asserts that *the product of the sines of perpendiculars from any point of a sphero-conic on two of its tangents is in a constant ratio to the square of the sine of the perpendicular from the same point on the arc of contact.*

In like manner the equation $\alpha\gamma = k\beta\delta$ asserts that *the product of the sines of the perpendiculars from any point of a sphero-conic on two opposite sides of an inscribed quadrilateral is in a constant ratio to the product of sines of perpendiculars on the other two sides.* And from this property again may be deduced, precisely as at *Conics*, Art. 259, that *the anharmonic ratio of the four arcs joining four fixed points on a sphero-conic to any other point on the curve is constant.* In like manner almost all the proofs of theorems respecting plane conics (given *Conics*, Chap. XIV.) apply equally to sphero-conics.

[All anharmonic properties of sphero-conics are particular cases of anharmonic properties of the (quartic) curve of intersection of any two quadrics. For a sphero-conic is the intersection of a quadric with a concentric sphere. Now the centre of a quadric is the pole of the line at infinity, therefore the sphere and quadric, projectively (Art. 144 (c)), are simply two quadrics the common centre being a vertex of the self-conjugate tetrahedron. E.g. If through a vertex of the common self-conjugate tetrahedron, four fixed lines are drawn meeting the quartic curve, the anharmonic ratio of the planes joining these four fixed lines to a variable line of the same kind is constant. This follows directly from the anharmonic property of conics, since the vertex is the vertex of a quadric cone passing through the quartic (Art. 136).]

245. If α, β represent the planes of circular section (or *cyclic planes*) of a cone, the equation of the cone is of the form $x^2 + y^2 + z^2 = k\alpha\beta$ (Art. 103), which, interpreted as in the last article, shows that the product of the sines of perpendiculars from any point of a sphero-conic on the two cyclic arcs is constant. Or, again, that, "*Given the base of a spherical triangle and the product of cosines of sides, the locus of vertex is a sphero-conic*, the cyclic arcs of which are the great circles having for their poles the extremities of the

given base." The form of the equation shows that the cyclic arcs of sphero-conics are analogous to the asymptotes of plane conics.

Every property of a sphero-conic can be doubled by considering the sphero-conic formed by the cone reciprocal to the given one. Thus (Art. 125) it was proved that the cyclic planes of one cone are perpendicular to the focal lines of the reciprocal cone. If then the points in which the focal lines meet the sphere be called the *foci of a sphero-conic*, the property established in this article proves that *the product of the sines of the perpendiculars let fall from the two foci on any tangent to a sphero-conic is constant.*

246. *If any great circle meet a sphero-conic in two points P, Q, and the cyclic arcs in points A, B, then $AP = BQ$.*

This is deduced from the property of the last article in the same way as the corresponding property of the plane hyperbola is proved. The ratio of the sines of the perpendiculars from P and Q on α is equal to the ratio of the sines of perpendiculars from Q and P on β. But the sines of the perpendiculars from P and Q on α are in the ratio $\sin AP : \sin AQ$, and therefore we have

$$\sin AP : \sin AQ :: \sin BQ : \sin BP,$$

whence it may easily be inferred that $AP = BQ$.

Reciprocally, the *two tangents from any point to a sphero-conic make equal angles with the arcs joining that point to the two foci.*

247. As a particular case of the theorem of Art. 246 we learn that *the portion of any tangent to a sphero-conic intercepted between the two cyclic arcs is bisected at the point of contact.* This theorem may also be obtained directly from the equation of a tangent, viz.

$$2(xx' + yy' + zz') = k(\alpha'\beta + \alpha\beta').$$

The form of this equation shows that the tangent at any point is constructed by joining that point to the intersection of its polar ($xx' + yy' + zz'$, see Art. 241) with $\alpha'\beta + \beta'\alpha$ which is the fourth harmonic to the cyclic arcs, α, β, and the line

joining the given point to their intersection. Since then the given point is 90° distant from its harmonic conjugate in respect of the two points where the tangent at that point meets the cyclic arcs, it is equidistant from these points (Art. 241).

Reciprocally, *the lines joining any point on a sphero-conic to the two foci make equal angles with the tangent at that point.*

248. From the fact that *the intercept by the cyclic arcs* on any tangent is bisected at the point of contact, it may at once be inferred by the method of infinitesimals (see *Conics*, Art. 396) that *every tangent to a sphero-conic forms with the cyclic arcs a triangle of constant area*, or a triangle the sum of whose base angles is constant. This may also be inferred trigonometrically from the fact that the product of sines of perpendiculars on the cyclic arcs is constant. For if we call the intercept on the tangent c, and the angles it makes with the cyclic arcs A and B, the sines of the perpendiculars on α and β are respectively $\sin \frac{1}{2}c \sin A$, $\sin \frac{1}{2}c \sin B$. But considering the triangle of which c is the base and A and B the base angles, then, by spherical trigonometry,

$$\sin^2\tfrac{1}{2}c \sin A \sin B = -\cos S \cos (S - C).$$

But C is given, therefore S, the half sum of the angles, is given.

Reciprocally, *the sum of the arcs joining the two foci to any point on a sphero-conic is constant.* Or the same may be deduced by the method of infinitesimals (see *Conics*, Art 392) from the theorem that the focal radii make equal angles with the tangent at any point.*

* Here, again, we can see that a sphero-conic may be regarded either as an ellipse or hyperbola. The focal lines each evidently meet the sphere in two diametrically opposite points. If we choose for foci two points within one of the closed curves in which the cone meets the sphere, then the *sum* of the focal distances is constant. But if we substitute for one of the focal distances FP, the focal distance from the diametrically opposite point, then since $F'P = 180° - FP$, we have the *difference* of the focal distances constant.

In like manner we may say that a variable tangent makes with the cyclic arcs angles whose difference is constant, if we substitute its supplement for one of the angles at the beginning of this article.

249. Conversely, again, we can *find the locus of a point on a sphere, such that the sum of its distances from two fixed points on the sphere may be constant.* The equation

$$\cos(\rho+\rho') = \cos a$$

may be written

$$\cos^2\rho + \cos^2\rho' - 2\cos\rho\cos\rho'\cos a = \sin^2 a.$$

If then α and β denote the planes which are the polars of the two given points, since we have $\alpha = \cos\rho$, the equation of the locus is

$$\alpha^2 + \beta^2 - 2\alpha\beta\cos a = \sin^2 a\,(x^2+y^2+z^2).$$

In order to prove that the planes α and β are perpendicular to focal lines of this cone, it is only necessary to show that sections parallel to either plane have a focus on the line perpendicular to it. Thus let α', α'' be two planes perpendicular to each other and to α, and therefore passing through the line which we want to prove a focal line. Then since

$$x^2+y^2+z^2 = \alpha^2 + \alpha'^2 + \alpha''^2,$$

the equation of the locus becomes

$$\sin^2 a\,(\alpha'^2+\alpha''^2) = (\beta - \alpha\cos a)^2.$$

If, then, this locus be cut by any plane parallel to α, $\alpha'^2+\alpha''^2$ is the square of the distance of a point on the section from the intersection of $\alpha'\alpha''$, and we see that this distance is in a constant ratio to the distance from the line in which $\beta - \alpha\cos a$ is cut by the same plane. This line is therefore the directrix of the section, the point $\alpha'\alpha''$ being the focus.

We see thus also that the general equation of a cone having the line xy for a focal line is of the form

$$x^2+y^2 = (ax+by+cz)^2;$$

whence again it follows that *the sine of the distance of any point on a sphero-conic from a focus is in a constant ratio to the sine of the distance of the same point from a certain directrix arc.*

[Ex. 1. Taking for axes the *G.C.* arc joining the two foci and the *G.C.* bisecting this at right angles, and for coordinates the *G.C.* perpendiculars on the axes (cf. Art. 240) the equation of a sphero conic may be written

$$\frac{\sin^2 x}{\sin^2 a} + \frac{\sin^2 y}{\sin^2 b} = 1,$$

where $\cos^2 b = \dfrac{\cos^2 a}{\cos^2 c}$. The *G.C.* distance between the foci is $2c$ and the sum

of the focal $G.C.$ distances is $2a$; we assume $2c < 180°$ and $a > c$. If $2c < 180°$ and $a < c$, the equation will be

$$\frac{\sin^2 x}{\sin^2 a} - \frac{\sin^2 y}{\sin^2 b} = 1, \text{ where } 1 + \sin^2 b = \frac{\cos^2 a}{\cos^2 c}.$$

Ex. 2. Prove that two confocal sphero-conics can be drawn through a point, and that one meets the y great circle in real, and the other in imaginary points. For one of those the sum of the focal distances (measured in the same direction) remains constant as we move along the same branch of the confocal, for the other the difference remains constant.

Ex. 3. Since confocal cones with a common vertex cut at right angles (Arts. 162, 176) the same is true of confocal sphero-conics.]

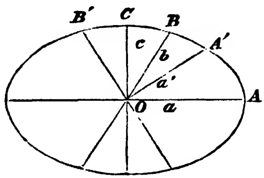
250. *Any two variable tangents meet the cyclic arcs in four points which lie on a circle.* For if L, M be two tangents and R the chord of contact, the equation of the sphero-conic may be written in the form $LM = R^2$; but this must be identical with $\alpha\beta = x^2 + y^2 + z^2$. Hence $\alpha\beta - LM$ is identical with $x^2 + y^2 + z^2 - R^2$. The latter quantity represents a small circle, having the same pole as R, and the form of the other shows that that circle circumscribes the quadrilateral $\alpha L\beta M$.

Reciprocally, the focal radii to any two points on a sphero-conic form a spherical quadrilateral in which a small circle can be inscribed. From this property, again, may be deduced the theorem that the sum or difference of the focal radii is constant, since the difference or sum of two opposite sides of such a quadrilateral is equal to the difference or sum of the remaining two.

251. From the properties just proved for cones can be deduced properties of quadrics in general. Thus *the product of the sines of the angles that any generator of a hyperboloid makes with the planes of circular section is constant.* For the generator is parallel to an edge of the asymptotic cone whose circular sections are the same as those of the surface. Again, since the focal lines of the asymptotic cone are the asymptotes of the focal hyperbola, it follows from Art. 248 that the sum or difference is constant of the angles which any generator of a hyperboloid makes with the asymptotes to the focal hyperbola. Again, *given one axis of a central section of a quadric, the sum or difference is given of the angles*

which its plane makes with the planes of circular section. For (Art. 102) given one axis of a central section its plane touches a cone concyclic with the given quadric, and therefore the present theorem follows at once from Art. 248.

We get an expression for the sum or difference of the angles, in terms of the given axis, by considering the principal section containing the greatest and least axes of the quadric. We obtain the cyclic planes by inflecting in that section, semi-diameters OB, OB' each $= b$. Then the planes containing these lines and perpendicular to the plane of the figure are the cyclic planes. Now if we draw any semi-diameter a' making the angle α with OC, we have



$$\frac{1}{a'^2} = \frac{\cos^2\alpha}{c^2} + \frac{\sin^2\alpha}{a^2}.$$

But a' is obviously an axis of the section which passes through it and is perpendicular to the plane of the figure, and (if a' be greater than b) α is evidently half the sum of the angles BOA', $B'OA'$ which the plane of the section makes with the cyclic planes. If a' be less than b, OA' falls between OB, OB', and α is half the difference of BOA', $B'OA'$. But this sum or difference is the same for all sections having the same axis. Hence, if a', b' be the axes of any central section, making angles θ, θ' with the cyclic planes, we have

$$\frac{1}{b'^2} = \frac{\cos^2\frac{1}{2}(\theta-\theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta-\theta')}{a^2},$$

$$\frac{1}{a'^2} = \frac{\cos^2\frac{1}{2}(\theta+\theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta+\theta')}{a^2}.$$

Subtracting, we have

$$\frac{1}{b'^2} - \frac{1}{a'^2} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin\theta \sin\theta',$$

or, *the difference of the squares of the reciprocals of the axes of a central section is proportional to the product of the sines of the angles it makes with the cyclic planes.*

252. We saw (Art. 246) that, given two sphero-conics having the same cyclic arcs, the intercept made by the outer on any tangent to the inner is bisected at the point of contact; and hence, by the method of infinitesimals, that tangent cuts off from the outer a segment of constant area (*Conics*, Art. 396).

Again, if two sphero-conics have the same foci, and if tangents be drawn to the inner from any point on the outer, these tangents are equally inclined to the tangent to the outer at that point. Hence, by infinitesimals (see *Conics*, Art. 399), *the excess of the sum of the two tangents over the included arc of the inner conic is constant.* This theorem is the reciprocal of the first theorem of this article, and it is so that it was obtained by Graves (see his Translation of Chasles's Memoir, p. 77).

253. *To find the locus of the intersection of two tangents to a sphero-conic which cut at right angles.* This is, in other words, to find the cone generated by the intersection of two rectangular tangent planes to a given cone $\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0$. Let the direction-angles of the perpendiculars to the two tangent planes be $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$; then they fulfil the relations

$$A\cos^2\alpha'+B\cos^2\beta'+C\cos^2\gamma'=0,$$
$$A\cos^2\alpha''+B\cos^2\beta''+C\cos^2\gamma''=0.$$

But if α, β, γ be the direction-cosines of the line perpendicular to both, we have $\cos^2\alpha=1-\cos^2\alpha'-\cos^2\alpha''$, &c. Therefore adding the two preceding equations, we have for the equation of the locus,

$$Ax^2+By^2+Cz^2=(A+B+C)(x^2+y^2+z^2),$$

a cone concyclic with the reciprocal of the given cone. Reciprocally, the envelope of a chord 90° in length is a sphero-conic, confocal with the reciprocal of the given cone.

254. *To find the locus of the foot of the perpendicular from the focus of a sphero-conic on the tangent.* The work of this question is precisely the same as that of the corresponding problem in plane conics, and the only difference is in the

interpretation of the result. Let the equation of the sphero-conic (Art. 249) be $x^2 + y^2 = t^2$ where $t = ax + by + cz$, then the equation of the tangent is

$$xx' + yy' = tt',$$

and of a perpendicular to it through the origin is

$$(x' - at')\,y - (y' - bt')\,x = 0.$$

Solving for x', y', and t' from these two equations, and substituting in $x'^2 + y'^2 = t'^2$, we get for the locus required,

$$(x^2 + y^2)\,\{(a^2 + b^2 - 1)\,(x^2 + y^2) + 2cz\,(ax + by) + c^2z^2\} = 0.$$

The quantity within the brackets denotes a cone whose circular sections are parallel to the plane z.

255. It may be inferred from Art. 242 that the quantity

$$\alpha \sin A + \beta \sin B + \gamma \sin C$$

has not, as *in plano*, a fixed value for the perpendiculars from any point. It remains then to ask how the three perpendiculars from any point on three fixed great circles are connected. But this question we have implicitly answered already, for the three perpendiculars are each the complement of one of the three distances from the three poles of the sides of the triangle of reference. If then a, b, c be the sides; A, B, C the angles of the triangle of reference, then α, β, γ the sines of the perpendiculars on the sides from any point are connected by the following relation, which is only a transformation of that of Art. 56.

$$\begin{aligned}&\alpha^2 \sin^2 A + \beta^2 \sin^2 B + \gamma^2 \sin^2 C + 2\beta\gamma \sin B \sin C \cos a\\ &+ 2\gamma\alpha \sin C \sin A \cos b + 2\alpha\beta \sin A \sin B \cos c\\ &= 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.\end{aligned}$$

The equation in this form represents a relation between the sines of the arcs represented by α, β, γ. If we want to get a relation between the perpendiculars from any point of the sphere on the planes represented by α, β, γ, we have evidently only to multiply the right-hand side of the preceding equation by r^2, and that equation in α, β, γ will be the transformation of the equation $x^2 + y^2 + z^2 = r^2$.

Hence, it appears that if we equate the left-hand side of

the preceding equation to zero, the equation will be the same as $x^2+y^2+z^2=0$, and therefore denotes the imaginary circle which is the intersection of two concentric spheres; that is to say, the imaginary circle at infinity (see Art. 139).

256. This equation may be used to find the *equation of the sphere inscribed in a given tetrahedron*, whose faces are $\alpha, \beta, \gamma, \delta$. If through the centre three planes be drawn parallel to α, β, γ, the perpendiculars on them from any point will be $\alpha-r, \beta-r, \gamma-r$. The equation of the sphere is therefore

$$(\alpha-r)^2 \sin^2 A + (\beta-r)^2 \sin^2 B + \&c.$$
$$= r^2 (1-\cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C).$$

But if L, M, N, P denote the areas of the four faces, we have $$L\alpha + M\beta + N\gamma + P\delta = (L+M+N+P)\,r.$$
Hence, by eliminating r, we arrive at a result reducible to the form of Art. 228.

257. The *equation of a small circle* (or right cone) is easily expressed. The sine of the distance of any point of the circle from the polar of the centre is constant. Hence, if α be that polar, the equation of the circle is $\alpha^2=\cos^2\rho\ (x^2+y^2+z^2)$.

All small circles then being given by equations of the form $S=\alpha^2$, their properties are all cases of those of conics having double contact with the same conic.

The theory of invariants may be applied to small circles. Let two circles S, S' be

$$x^2+y^2+z^2-\alpha^2\sec^2\rho,\ x^2+y^2+z^2-\beta^2\sec^2\rho',$$

and let us form the condition that $\lambda S+S'$ should break up into factors. This cubic being

$$\lambda^3\Delta+\lambda^2\Theta+\lambda\Theta'+\Delta'=0,$$

we have $$\Delta=-\tan^2\rho,\ \Delta'=-\tan^2\rho',$$
$$\Theta=\sec^2\rho\sec^2\rho'\sin^2 D-2\tan^2\rho-\tan^2\rho',$$
$$\Theta'=\sec^2\rho\sec^2\rho'\sin^2 D-2\tan^2\rho'-\tan^2\rho,$$

where D is the distance between the centres.

Now the corresponding values for two circles in a plane are

$$\Delta=-r^2,\ \Delta'=-r'^2,\ \Theta=D^2-2r^2-r'^2,\ \Theta'=D^2-2r'^2-r^2.$$

Hence, if any invariant relation between two circles in a

plane is expressed as a function of the radii and of the distance between their centres, the corresponding relation for circles on a sphere is obtained by substituting for r, r', D; $\tan\rho$, $\tan\rho'$, and $\sec\rho\sec\rho'\sin D$.

Thus the condition that two circles in a plane should touch is obtained by forming the discriminant of the cubic equation, and is either $D=0$ or $D=r\pm r'$. The corresponding equation therefore for two circles on a sphere is

$$\tan\rho\pm\tan\rho'=\sec\rho\sec\rho'\sin D, \text{ or } \sin D=\sin(\rho\pm\rho').$$

Again, if two circles in a plane be the one inscribed in, the other circumscribed about, the same triangle, the invariant relation is fulfilled $\Theta^2=4\Delta\Theta'$, which gives for the distance between their centres the expression $D^2=R^2-2Rr$.

The distance therefore between the centres of the inscribed and circumscribed circles of a spherical triangle is given by the formula

$$\sec^2 P\sec^2\rho\sin^2 D=\tan^2 P-2\tan P\tan\rho.$$

So, in like manner, we can get the relation between two circles inscribed in, and circumscribed about, the same spherical polygon.

258. The *equation of any small circle* (*or right cone*) *in trilinear coordinates* must (Art. 255) be of the form

$$\alpha^2\sin^2 A+\beta^2\sin^2 B+\gamma^2\sin^2 C$$
$$+2\beta\gamma\sin B\sin C\cos a+2\gamma\alpha\sin C\sin A\cos b$$
$$+2\alpha\beta\sin A\sin B\cos c=(l\alpha+m\beta+n\gamma)^2.$$

If now the *small circle circumscribe the triangle* $\alpha\beta\gamma$, the coefficients of α^2, β^2, and γ^2 must vanish, we must therefore have $l\alpha+m\beta+n\gamma=\alpha\sin A+\beta\sin B+\gamma\sin C$. Hence, as was proved before, this represents the polar of the centre of the circumscribing circle. Substituting the values $\sin A$, $\sin B$, $\sin C$ for l, m, n, the equation of the small circle becomes

$$\beta\gamma\tan\tfrac{1}{2}a+\gamma\alpha\tan\tfrac{1}{2}b+\alpha\beta\tan\tfrac{1}{2}c=0.$$

The *equation of the inscribed circle* turns out to be of exactly the same form as in the case of plane triangles, viz.

$$\cos\tfrac{1}{2}A\sqrt{(\alpha)}\pm\cos\tfrac{1}{2}B\sqrt{(\beta)}\pm\cos\tfrac{1}{2}C\sqrt{(\gamma)}=0.$$

The tangential equation of a small circle may either be derived

by forming the reciprocal of that given at the commencement of this article, or directly from Art. 243, by expressing that the perpendicular from the centre on $\lambda\alpha + \mu\beta + \nu\gamma$ is constant. We find thus for the tangential equation of the circle whose centre is $\alpha'\beta'\gamma'$ and radius ρ

$$\sin^2\rho\ (\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C) = (\alpha'\lambda + \beta'\mu + \gamma'\nu)^2;$$

a form also showing (see Art. 257) that every circle has double contact with the imaginary circle at infinity.

259. As a concluding exercise on the formulæ of this chapter, we investigate Hart's extension of Feuerbach's theorem for plane triangles, viz. that *the four circles which touch the sides are all touched by the same circle.*

It is easier to work with the tangential equations. The tangential equations of circles which touch the sides of the triangle of reference must want the terms λ^2, μ^2, ν^2, and therefore evidently are

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C = (\lambda \pm \mu \pm \nu)^2;$$

or

$$\mu\nu \cos^2 \tfrac{1}{2}A + \nu\lambda \cos^2 \tfrac{1}{2}B + \lambda\mu \cos^2 \tfrac{1}{2}C = 0 \ldots\ldots (1),$$

$$\mu\nu \cos^2 \tfrac{1}{2}A - \nu\lambda \sin^2 \tfrac{1}{2}B - \lambda\mu \sin^2 \tfrac{1}{2}C = 0 \ldots\ldots (2),$$

$$-\mu\nu \sin^2 \tfrac{1}{2}A + \nu\lambda \cos^2 \tfrac{1}{2}B - \lambda\mu \sin^2 \tfrac{1}{2}C = 0 \ldots\ldots (3),$$

$$-\mu\nu \sin^2 \tfrac{1}{2}A - \nu\lambda \sin^2 \tfrac{1}{2}B + \lambda\mu \cos^2 \tfrac{1}{2}C = 0 \ldots\ldots (4),$$

all of which four are touched by the circle

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C = \{\lambda \cos (B - C) + \mu \cos (C - A) + \nu \cos (A - B)\}^2 \ldots\ldots (5).$$

For the centres of similitude of the circles (1) and (5) are given by the tangential equations

$$(\lambda + \mu + \nu) \pm \{\lambda \cos (B - C) + \mu \cos (C - A) + \nu \cos (A - B)\} = 0,$$

one of them therefore is

$$\lambda \sin^2 \tfrac{1}{2} (B - C) + \mu \sin^2 \tfrac{1}{2} (C - A) + \nu \sin^2 \tfrac{1}{2} (A - B).$$

And (*Conics*, Art. 127) the condition that this point should be on the circle (1) is

$$\cos \tfrac{1}{2}A \sin \tfrac{1}{2} (B - C) + \cos \tfrac{1}{2}B \sin \tfrac{1}{2} (C - A) + \cos \tfrac{1}{2}C \sin \tfrac{1}{2} (A - B) = 0,$$

which is satisfied. The coordinates of the point of contact are accordingly

$$\sin^2 \tfrac{1}{2} (B - C),\ \sin^2 \tfrac{1}{2} (C - A),\ \sin^2 \tfrac{1}{2} (A - B).$$

It is proved, in like manner, that the circle (5) touches the three other circles.

260. The coordinates of the centre of Hart's circle have been proved to be $\cos(B-C)$, $\cos(C-A)$, $\cos(A-B)$. This point therefore lies on the line joining the point whose coordinates are $\cos B\cos C$, $\cos C\cos A$, $\cos A\cos B$ to the point whose coordinates are $\sin B\sin C$, $\sin C\sin A$, $\sin A\sin B$; that is to say (Art. 242) on the line joining the intersection of perpendiculars to the intersection of bisectors of sides. Since

$$\cos(B-C)-\cos A=2\sin\tfrac{1}{2}(A+B-C)\sin\tfrac{1}{2}(C+A-B);$$

the centre lies also on the line joining the point $\cos A$, $\cos B$, $\cos C$ to the point

$$\sin(S-B)\sin(S-C),\ \sin(S-C)\sin(S-A),\ \sin(S-A)\sin(S-B).$$

The first point is the intersection of lines drawn through each vertex making the same angle with one side that the perpendicular makes with the other; the second point is the intersection of perpendiculars let fall from each vertex on the line joining the middle points of the adjacent sides. The centre of Hart's circle is thus constructed as the intersection of two known lines.

261. The problem might also have been investigated by the direct equation. We write $a\sin A=x$, &c., so that the equation of the imaginary circle at infinity is $U=0$, where

$$U=x^2+y^2+z^2+2yz\cos a+2zx\cos b+2xy\cos c.$$

Then the equation of the inscribed circle is

$$U=\{x\cos(s-a)+y\cos(s-b)+z\cos(s-c)\}^2$$

where $2s=a+b+c$. For this equation expanded is

$$x^2\sin^2(s-a)+y^2\sin^2(s-b)+z^2\sin^2(s-c)$$
$$-2yz\sin(s-b)\sin(s-c)-2zx\sin(s-c)\sin(s-a)$$
$$-2xy\sin(s-a)\sin(s-b)=0.$$

U is not altered if we change the sign of either a, b, or c. Consequently we get three other circles also touching x, y, z if we change the signs of either a, b, or c in the equation of the inscribed circle. All four circles will be touched by

$$U=\left\{\frac{x\cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a}+\frac{y\cos\frac{1}{2}c\cos\frac{1}{2}a}{\cos\frac{1}{2}b}+\frac{z\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}\right\}^2.$$

This last equation not being altered by changing the sign of a, b, or c, it is evident that if it touches one it touches all. Now one of its common chords with the inscribed circle is

$$x\left\{\cos(s-a)-\frac{\cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a}\right\}+y\left\{\cos(s-b)-\frac{\cos\frac{1}{2}c\cos\frac{1}{2}a}{\cos\frac{1}{2}b}\right\}$$
$$+z\left\{\cos(s-c)-\frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}\right\},$$

which reduced is

$$\frac{x}{\sin(s-b)-\sin(s-c)}+\frac{y}{\sin(s-c)-\sin(s-a)}$$
$$+\frac{z}{\sin(s-a)-\sin(s-b)}=0.$$

But the condition that the line $Ax+By+Cz$ shall touch $\sqrt{(ax)}+\sqrt{(by)}+\sqrt{(cz)}$ is $\frac{a}{A}+\frac{b}{B}+\frac{c}{C}$. Applying this condition the line we are considering will touch the inscribed circle if

$$\sin(s-a)\{\sin(s-b)-\sin(s-c)\}+\sin(s-b)\{\sin(s-c)$$
$$-\sin(s-a)\}+\sin(s-c)\{\sin(s-a)-\sin(s-b)\}=0;$$

a condition which is evidently fulfilled. It will be seen that the condition is also fulfilled that the common tangent in question should touch $\sqrt{(x)}+\sqrt{(y)}+\sqrt{(z)}$; that is to say, the sphero-conic which touches at the middle points of the sides, a fact remarked by Hamilton, and which leads at once to a construction for that tangent as the fourth common tangent to two conics which have three known tangents common.

The polar of the centre of Hart's circle has been thus proved to be

$$\alpha\sin A\frac{\cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a}+\beta\sin B\frac{\cos\frac{1}{2}c\cos\frac{1}{2}a}{\cos\frac{1}{2}b}$$
$$+\gamma\sin C\frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}=0,$$

or $$\alpha\tan\tfrac{1}{2}a+\beta\tan\tfrac{1}{2}b+\gamma\tan\tfrac{1}{2}c=0,$$

which may be also written

$$\alpha\cos(S-A)+\beta\cos(S-B)+\gamma\cos(S-C)=0$$

forms which lead to other constructions for the centre of this circle.

The radius of the circle touching three others whose centres are known, and whose radii are r, r', r'' may be determined by substituting $r+R$, $r'+R$, $r''+R$ for d, e, f in the formulæ of Arts. 54, 56, and solving for R. Applying this method to the three escribed circles I have found that *the tangent of the radius of Hart's circle is half the tangent of the radius of the circumscribing circle of the triangle.*

[Hart's theorem is a particular case of a more general property of quadrics (cf. Art. 139). The sides of a triangle ABC formed by three conics lying on a quadric whose planes are concurrent at P are touched by four conics lying on the quadric and these are touched by another conic lying on the quadric. Strictly speaking the conics BC, CA, AB form eight triangles, and we have thus eight conics touching all three. The triangles fall into opposite pairs with reference to the point of intersection of the planes of the original three conics as in the case of the sphere. The theorem may be deduced from Hart's by projection (Art. 144c); for any quadric may be projected into a sphere, and conics on the quadric, being plane sections, project into small circles on the sphere. Also the projection can be chosen so that the point P projects into the centre of the sphere. Or the theorem may be proved directly by generalizing the method of Art. 259, by using the tangential equations of cones with vertices at P, the origin.]

CHAPTER XI.

GENERAL THEORY OF SURFACES. (INTRODUCTORY CHAPTER) —CURVATURE.

Characteristics of Surfaces.

262. RESERVING for a future chapter a more detailed examination of the properties of surfaces in general, we shall in this chapter give an account of such parts of the general theory as can be obtained with least trouble.

Let the general equation of a surface be written in the form

$$\begin{gathered} A \\ + Bx + Cy + Dz \\ + Ex^2 + Fy^2 + Gz^2 + 2Hyz + 2Kzx + 2Lxy \\ + \&c. = 0, \end{gathered}$$

or, as we shall write it often for shortness,

$$u_0 + u_1 + u_2 + u_3 + \&c. = 0,$$

where u_2 means the aggregate of terms of the second degree, &c. Then it is evident that u_0 consists of one term, u_1 of three, u_2 of six, &c. The total number of terms in the equation is therefore the sum of $n+1$ terms of the series, 1, 3, 6, 10, &c., that is to say, $\dfrac{(n+1)\ (n+2)\ (n+3)}{1.2.3.}$.

The number of conditions necessary to determine a surface of the n^{th} degree is one less than this, or $= \dfrac{n(n^2+6n+11)}{6}$.

The equation above written can be thrown into the form of a polar equation by writing $\rho \cos \alpha$, $\rho \cos \beta$, $\rho \cos \gamma$ for x, y, z, when we obviously obtain an equation of the n^{th} degree, which will determine n values of the radius vector answering to any assigned values of the direction-angles α, β, γ

[Ex. 1. If p be the number of points necessary to determine a surface of n^{th} degree, then all surfaces of the n^{th} degree passing through $p-1$ fixed points pass, in general, through a fixed curve. They form a "pencil" of surfaces of the form $U + \lambda V = 0$.

Ex. 2. Surfaces of the n^{th} degree passing through $p-2$ fixed points pass through $n^3 - p + 2$ other fixed points, and form a "sheaf" of surfaces of the form $\lambda U + \mu V + \nu W = 0$.]

263. If now the origin be on the surface, we have $u_0 = 0$, and one of the roots of the equation is always $\rho = 0$. But a second root of the equation will be $\rho = 0$ if a, β, γ be connected by the relation

$$B \cos a + C \cos \beta + D \cos \gamma = 0.$$

Now multiplying this equation by ρ it becomes

$$Bx + Cy + Dz = 0,$$

and we see that it expresses merely that the radius vector must lie in the plane $u_1 = 0$. No other condition is necessary in order that the radius should meet the surface in two coincident points. Thus we see that in general *through an assumed point on a surface we can draw an infinity of radii vectores which will there meet the surface in two coincident points; that is to say, an infinity of tangent lines to the surface; and these lines lie all in one plane, called the tangent plane, determined by the equation* $u_1 = 0$.

264. *The section of any surface made by a tangent plane is a curve having the point of contact for a double point.**

Every radius vector to the surface, which lies in the tangent plane, is of course also a radius vector to the section made by that plane; and since every such radius vector (Art. 263) meets the section at the origin in two coincident points, the origin is, by definition, a double point (see *Higher Plane Curves*, Art. 37).

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* I had supposed that this remark was first made by Cayley: Gregory's *Solid Geometry*, p. 132. I am informed, however, by Professor Cremona that the point had been previously noticed by the Italian geometer, Bedetti, in a memoir read before the Academy of Bologna, 1841. The theorem is a particular case of that of Art. 203. Observe that the tangents at the double point are the inflexional tangents of Art. 265, and that these may be considered as identical with the asymptotes of the indicatrix, Art. 266. There is thus an anticipation of the theorem by Dupin (1813).

We have already had an illustration of this in the case of hyperboloids of one sheet, which are met by any tangent plane in a conic having a double point, that is to say, in two right lines. And the point of contact of the tangent plane to a quadric of any other species is equally to be considered as the intersection of two imaginary right lines.

From this article it follows conversely, that *any plane meeting a surface in a curve having a double point touches the surface, the double point being the point of contact.* If the section have two double points, the plane will be a double tangent plane; and if it have three double points, the plane will be a triple tangent plane. Since the equation of a plane contains three constants, it is possible to determine a plane which will satisfy any three conditions, and therefore a finite number of planes can in general be determined which will meet a given surface in a curve having three double points: that is to say, *a surface has in general a determinate number of triple tangent planes.* It will also have *an infinity of double tangent planes*, the points of contact lying on a certain curve locus on the surface. The degree of this curve, and the number of triple tangent planes will be subjects of investigation hereafter.

265. *Through an assumed point on a surface it is generally possible to draw two lines which shall there meet the surface in three coincident points, and these are the tangent lines at the double point on the section by the tangent plane.*

In order that the radius vector may meet the surface in three coincident points, we must not only, as in Art. 263, have the condition fulfilled

$$B\cos\alpha + C\cos\beta + D\cos\gamma = 0,$$

but also $E\cos^2\alpha + F\cos^2\beta + G\cos^2\gamma$

$$+ 2H\cos\beta\cos\gamma + 2K\cos\gamma\cos\alpha + 2L\cos\alpha\cos\beta = 0.$$

For if these conditions were fulfilled, A being already supposed to vanish, the equation of the n^{th} degree which determines ρ, becomes divisible by ρ^3, and has therefore three roots $= 0$. The first condition expresses that the radius vector must lie

in the tangent plane u_1. The second expresses that the radius vector must lie in the surface $u_2 = 0$, or

$$Ex^2 + Fy^2 + Gz^2 + 2Hyz + 2Kzx + 2Lxy = 0.$$

This surface is a cone of the second degree (Art. 66) and since every such cone is met by a plane passing through its vertex in two right lines, two right lines can be found to fulfil the required conditions.

Every plane (other than the tangent plane) drawn through either of these lines meets the surface in a section having the point of contact for a point of inflexion. For a point of inflexion is a point, the tangent at which meets the curve in three coincident points (*Higher Plane Curves*, Art. 46). On this account we shall call the two lines which meet the surface in three coincident points, the *inflexional* tangents at the point.*

The existence of these two lines may be otherwise perceived thus. We have proved that the point of contact is a double point in the section made by the tangent plane. And it has been proved (*Higher Plane Curves*, Art. 37) that at a double point can always be drawn two lines meeting the section (and therefore the surface) in three coincident points.

266. A double point may be one of three different kinds, according as the tangents at it are real, coincident, or imaginary. Accordingly the *contact of a plane with a surface may be of three kinds according as the tangent plane meets it in a section having a node, a cusp, or a conjugate point; or*, in other words, *according as the inflexional tangents are real, coincident, or imaginary.*

If instead of the tangent plane we consider with Dupin, a parallel plane indefinitely near thereto, the section of the surface by this plane may, in the neighbourhood of the point, be regarded as a curve of the second order, which may be an ellipse, hyperbola, or a *pair of parallel lines;* this curve of

* They are called by German writers the "Haupt-tangenten".

the second order is called the *Indicatrix*.* Analytically, if taking the given point of the surface for origin, we take the normal for the axis of z, and the axes of x, y in the tangent plane; then considering x, y as infinitesimals of the first order, and consequently z as an infinitesimal of the second order, the equation of the surface, regarding z as a given constant, gives the equation of the section, and if herein we neglect infinitesimals of an order superior to the second, this reduces itself to an equation of the form

$$z + ax^2 + 2hxy + by^2 = 0,$$

an equation of the second order representing the indicatrix; viz. according as $ab - h^2$ is positive, negative, or zero, this is an ellipse, hyperbola or pair of parallel lines.† Geometrically, the section of the surface is either a closed curve, such as the ellipse; or, attending only to the curve in the neighbourhood of the given point, it consists of two arcs having their convexities turned towards each other, and which may be considered as portions of the two branches of a hyperbola; or the convexity vanishes, and the arcs are infinitesimal portions of two parallel right lines.

[The position of a hyperbolic indicatrix is ambiguous, since we get hyperbolas of different aspect according as the parallel plane is drawn on one or the other side of the tangent plane. But for the purposes for which the indicatrix is used this makes no difference since the hyperbolas are similar, and their orthogonal projections on the tangent plane are similarly situated.]

If points on a surface be called *elliptic*, *hyperbolic*, or *parabolic*, according to the nature of the indicatrix, we shall presently show that in general parabolic points form a curve

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* Dupin, see the *Développements de Géométrie* (1813), p. 49, is quite correct, he says: "En général, une courbe du second degré, dont le centre P nous est donné, ne peut être qu'une ellipse ou une hyperbole. Elle peut cependant être une parabole: alors elle se présente sous la forme de deux lignes droites parallèles *équidistantes de leur centre*."

† This is sometimes expressed as follows: When the plane of xy is the tangent plane, and the equation of the surface is expressed in the form $z = \phi\ (x, y)$, we have an elliptic, hyperbolic, or parabolic point, according as $\left(\frac{d^2z}{dx\,dy}\right)^2$ is less than, greater than, or equal to $\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right)$. It will be easily seen that this is equivalent to the statement in the text.

locus on the surface, this curve separating the elliptic from the hyperbolic points.

[A surface is often described as *synclastic* where its points are elliptic, and *anticlastic* where its points are hyperbolic. In an anticlastic region neighbouring points on a surface lie on both sides of the tangent plane at the point, in a synclastic region they lie only on one side. The ellipsoid, the hyperboloid of two sheets, and the elliptic paraboloid are everywhere synclastic; the hyperboloid of one sheet, and the hyperbolic paraboloid are everywhere anticlastic; the cone and the cylinder (of any degree) consist of parabolic points, but the vertex of the cone being a singular point is not to be classified in this way.]

In the case of a surface of the second order, taking the axes as above, the equation of the surface is

$$z + ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0,$$

which equation, if we regard therein x and y as infinitesimals of the first order, and therefore z as infinitesimal of the second order, reduces itself to $z + ax^2 + 2hxy + by^2 = 0$, viz. z being regarded as a constant, this is an equation of the form already mentioned as that of the indicatrix for a surface of any order whatever. The original equation, regarding therein z as a given constant, is the equation of the section of the surface by a plane parallel to the tangent plane, but it is not the proper equation of the indicatrix. To further explain this, suppose that the surface were of the third or any higher order, then besides the terms written down, there would have been in the equation terms $(x, y)^3$, &c.; to obtain the indicatrix as a curve of the second order, we must of necessity neglect these terms of the third order, and there is therefore no meaning in taking into account the terms $2gxz = 2fyz$ also of the third order, or the term cz^2 which is of the fourth order.*

In the case where the indicatrix is a hyperbola, then supposing the parallel plane to move parallel to itself until it coincides with the tangent plane, this hyperbola becomes a pair of real lines; viz. these are the inflexional tangents of Art. 265. And generally *the two inflexional tangents may be regarded as the asymptotes (real or imaginary) of the*

* See *Messenger of Mathematics*, Vol. V. (1870), p. 187.

indicatrix considered as lying *in* [or as orthogonally projected on] the tangent plane; they have been on this account termed the *asymptotic lines* of the point of the surface. If from any point of the surface we pass along one of these lines to a consecutive point, and thence along the consecutive line to a second point on the surface, and so on, we obtain a curve; and we have thus on the surface two series of curves, which are the *asymptotic curves*. In the case of a quadric surface, these are the two series of right lines on the surface.

[At a parabolic point the asymptotic lines coincide; they have the common limiting positions of the two parallel lines which represent, in the neighbourhood of the point, the form of the section in which the plane parallel to the tangent plane meets the surface as the former plane moves towards coincidence with the latter. A parabolic point is the limiting case of an elliptic point, when the ratio of the minor to the major axis of the elliptic indicatrix becomes zero; it is also the limiting case of a hyperbolic point when the angle between the asymptotes of the hyperbolic indicatrix becomes zero.

The *anchor ring* illustrates the three kinds of points. It is a surface generated by a circle revolving round an axis in its plane, the axis not cutting the circle. The inner portion of the surface is anticlastic, the outer portion is synclastic, and the parabolic curve consists of the two circles separating these portions.]

267. Knowing the equation of the tangent plane when the origin is on the surface, we can, by transformation of coordinates, find the *equation of the tangent plane at any point*. It is proved, precisely as at Art. 62, that this equation may be written in either of the forms

$$(x-x')\frac{dU'}{dx'}+(y-y')\frac{dU'}{dy'}+(z-z')\frac{dU'}{dz'}=0,$$

or

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+w\frac{dU'}{dw'}=0.$$

268. Let it be required now to find the tangent plane at a point, indefinitely near the origin, on the surface

$$z+ax^2+2hxy+by^2+2gxz+2fyz+cz^2+\&c.=0.$$

We have to suppose x', y' so small that their squares may be neglected; while, since the consecutive point is on the tangent plane, we have $z'=0$; or more accurately, the equation of the surface shows that z' is a quantity of the same order as the squares of x' and y'. Then, either by the formula of the

last article, or else directly by putting $x+x'$, $y+y'$ for x and y, and taking the linear part of the transformed equation, the equation of a consecutive tangent plane is found to be

$$z+2\,(ax'+hy')\,x+2\,(hx'+by')y=0.$$

Now (see *Conics*, Art. 141) $(ax'+hy')\,x+(hx'+by')y$ denotes the diameter of the conic $ax^2+2hxy+by^2=1$, which is conjugate to that to the point $x'y'$. Hence *any tangent plane is intersected by a consecutive tangent plane in the diameter of the indicatrix conjugate to the direction in which the consecutive point is taken.*

This, in fact, is geometrically evident from Dupin's point of view. For if we admit that the points consecutive to the given one lie on an infinitely small conic, we see that the tangent plane at any of them will pass through the tangent line to that conic; and this tangent line ultimately coincides with the diameter conjugate to that drawn to the point of contact; for the tangent line is parallel to this conjugate diameter and infinitely close to it.

Thus, then, all the tangent lines which can be drawn at a point on a surface may be distributed into pairs, such that the tangent plane at a consecutive point on either will pass through the other. Two tangent lines so related are called *conjugate tangents.*

In the case where the two inflexional tangents are real, the relation between two conjugate tangents may be otherwise stated. Take the inflexional tangents for the axes of x and y, which is equivalent to making a and $b=0$ in the preceding equation; then the equation of a consecutive tangent plane is $z+2h\,(x'y+y'x)=0$. And since the lines x, y, $x'y+y'x$, $x'y-y'x$ form a harmonic pencil, we learn that *a pair of conjugate tangents form, with the inflexional tangents, a harmonic pencil.* This is in fact the theorem that a pair of conjugate diameters of a conic are harmonics in regard to the asymptotes.

269. In the case where the origin is a parabolic point, the equation of the surface can be thrown into the form

$z + ay^2 + \&c. = 0$, and the equation of a consecutive tangent plane will be $z + 2ay'y = 0$. Hence the tangent plane at *every* point consecutive to a parabolic point passes through the inflexional tangent; and if the consecutive point be taken in this direction, so as to have $y' = 0$, then the consecutive tangent plane coincides with the given one. Hence *the tangent plane at a parabolic point is to be considered as a double tangent plane*, since it touches the surface at two consecutive points.* In this way parabolic points on surfaces may be considered as analogous to points of inflexion on plane curves: for we have proved (*Higher Plane Curves*, Art. 46) that the tangent line at a point of inflexion is in like manner to be regarded as a double tangent. A further analogy between parabolic points and points of inflexion will be afterwards stated.

It is necessary to have a name to distinguish double tangent planes which touch in two distinct points, from those now under consideration, where the two points of contact coincide. We shall therefore call the latter *stationary tangent planes*, the word expressing that the tangent plane being supposed to move round as we pass from one point of the surface to another, in this case remains for an instant in the same position [provided we move along the inflexional tangent]. For the same reason we have called the tangent lines at points of inflexion in plane curves, stationary tangents.

270. If on transforming the equation to any point on a surface as origin we have not only $u_0 = 0$, but also all the terms in $u_1 = 0$, so that the equation takes the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + u_3 + \&c. = 0,$$

then it is easy to see, in like manner, that *every* line through the origin meets the curve in two coincident points; and the origin is then called a double or *conical* point. It is easy to see also that a line through the origin there meets the surface

* I believe this was first pointed out in a paper of mine, *Cambridge and Dublin Mathematical Journal*, Vol. III., p. 45.

in *three* coincident points, provided that its direction-cosines satisfy the equation

$$a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma$$
$$+ 2f \cos \beta \cos \gamma + 2g \cos \gamma \cos \alpha + 2h \cos \alpha \cos \beta = 0.$$

In other words, *through a conical point on a surface can be drawn an infinity of lines which will meet the surface in three coincident points, and these will all lie on a cone of the second degree* whose equation is $u_2 = 0$. Further, of these lines *six will meet the surface in four coincident points;* namely, the lines of intersection of the cone u_2 with the cone of the third degree $u_3 = 0$.

Double points on surfaces might be classified according to the number of these lines which are real, or according as two or more of them coincide, but we shall not enter into these details. The only special case which it is important to mention is when the cone u_2 resolves itself into two planes [a *bi-planar* double point]; and this again includes the still more special case when these two planes coincide [a *uni-planar* double point]; that is to say, when u_2 is a perfect square.

271. Every plane drawn through a conical point may, in one sense, be regarded as a tangent plane to the surface, since it meets the surface in a section having a double point, but in a special sense the tangent planes to the cone u_2 are to be regarded as tangent planes to the surface, and the sections of the surface by these planes will each have the origin as a cusp. *To a conical point, then, on a surface (which is a point at which can be drawn an infinity of tangent planes), will in general correspond on the reciprocal surface a plane touching the surface in an infinity of points, which will in general lie on a conic.* If, however, the cone u_2 resolves itself into two planes, the point is in the strict sense a double point, and there corresponds to it on the reciprocal surface a double tangent plane having two points of contact.

272. The results obtained in the preceding articles, by

taking as our origin the point we are discussing, we shall now extend to the case where the point has any position whatever. Let us first remind the reader (see Art. 49) that since the equations of a right line contain four constants, a finite number of right lines can be determined to fulfil four conditions (as, for instance, to touch a surface four times), while an infinity of lines can be found to satisfy three conditions (as, for instance, to touch a surface three times), these right lines generating a certain surface, and their points of contact lying on a certain locus. In a subsequent chapter we shall return to the problem to determine in general the number of solutions when four conditions are given, and to determine the degree of the surface generated, and of the locus of points of contact, when three conditions are given. In this chapter we confine ourselves to the case when the right line is required to pass through a given point, whether on the surface or not. This is equivalent to two conditions; and an infinity of right lines (forming a cone) can be drawn to satisfy one other condition, while a finite number of right lines can be drawn to satisfy two other conditions.

We use Joachimsthal's method employed, *Conics*, Art. 290, *Higher Plane Curves*, Art. 59, and Art. 75 of this volume. If the quadri-planar coordinates of two points be $x'y'z'w'$, $x''y''z''w''$, then the points in which the line joining them is cut by the surface are found by substituting in the equation of the surface, for x, $\lambda x' + \mu x''$, for y, $\lambda y' + \mu y''$, &c. The result will give an equation of the n^{th} degree in $\lambda : \mu$, whose roots will be the ratios of the segments in which the line joining the two given points is cut by the surface at any of the points where it meets it. And the coordinates of any of the points of meeting are $\lambda' x' + \mu' x''$, $\lambda' y' + \mu' y''$, $\lambda' z' + \mu' z''$, $\lambda' w' + \mu' w''$, where $\lambda' : \mu'$ is one of the roots of the equation of the n^{th} degree. All this will present no difficulty to any reader who has mastered the corresponding theory for plane curves. And, as in plane curves, the result of the substitution in question may be written

$$\lambda^n U' + \lambda^{n-1}\mu\Delta U' + \tfrac{1}{2}\lambda^{n-2}\mu^2\Delta^2 U' + \&c. = 0$$

where Δ represents the operation

$$x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}+w\frac{d}{dw'}.$$

Following the analogy of plane curves we shall call the surface represented by

$$x'U_1+y'U_2+z'U_3+w'U_4=0,*$$

the *first polar* of the point $x'y'z'w'$. We shall call

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}+w'\frac{d}{dw}\right)^2 U=0,$$

the *second polar*, and so on; the *polar plane* of the same point being

$$xU_1'+yU_2'+zU_3'+wU_4'=0.$$

Each polar surface is manifestly also a polar of the point $x'y'z'w'$ with regard to all the other polars of higher degree.

If a point be on a surface all its polars touch the tangent plane at that point; for the polar plane with regard to the surface is the tangent plane; and this must also be the polar plane with regard to the several polar surfaces. This may also be seen by taking the polar of the origin with regard to

$$u_0w^n+u_1w^{n-1}+u_2w^{n-2}+\&c.,$$

where we have made the equation homogeneous by the introduction of a new variable w. The polar surfaces of the origin are got by differentiating with regard to this new variable. Thus the first polar is

$$nu_0w^{n-1}+(n-1)u_1w^{n-2}+(n-2)u_2w^{n-3}+\&c.,$$

and if $u_0=0$, the terms of the first degree, both in the surface and in the polar, will be u_1.

[**Ex.** The inflexional tangents of a surface at any point coincide with those of the polar surfaces of the point (see Art. 265).]

273. If now the point $x'y'z'w'$ be *on* the surface U' vanishes, and one of the roots of the equation in $\lambda : \mu$ will be $\mu=0$. A second root of that equation will be $\mu=0$, and the line will meet the surface in two coincident points at the point $x'y'z'w'$, provided that the coefficient of $\lambda^{n-1}\mu$ vanish in

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* As at Art. 59, U_1, U_2, U_3, U_4 denote half the differential coefficients of U with regard to x, y, z, w.

the equation referred to. And in order that this should be the case, it is manifestly sufficient that $x''y''z''w''$ should satisfy the equation of the plane

$$xU_1' + yU_2' + zU_3' + wU_4' = 0.$$

It is proved, then, that all the tangent lines to a surface which can be drawn at a given point lie in a plane whose equation is that just written. By subtracting from this equation, the identity

$$x'U_1' + y'U_2' + z'U_3' + w'U_4' = 0,$$

we get the ordinary Cartesian equation of the tangent plane, viz.

$$(x - x')\ U_1' + (y - y')\ U_2' + (z - z')\ U_3' = 0.$$

Hence, again, by Art. 45, can immediately be deduced the equations of the normal, viz.

$$\frac{x - x'}{U_1'} = \frac{y - y'}{U_2'} = \frac{z - z'}{U_3'}.$$

274. The right line will meet the surface in three consecutive points, or the equation we are considering will have for three of its roots $\mu = 0$, if not only the coefficients of λ and $\lambda^{n-1}\mu$ vanish, but also that of $\lambda^{n-2}\mu^2$; that is to say, if the line we are considering not only lies in the tangent plane, but also in the polar quadric,

$$\left(x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'} + w\frac{d}{dw'}\right)^2 U' = 0.$$

Now (Art. 272) when a point is on a surface all its polars touch the surface. The *tangent plane therefore, touching the polar quadric, meets it in two right lines, real or imaginary, which are the two inflexional tangents to the surface* (Art. 265).*

[Ex. The coordinates being Cartesian and rectangular (x, y, z), *a point on the surface is elliptic, hyperbolic or parabolic, according as the determinant*

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

is positive, negative or zero, where

$$a = \frac{d^2U}{dx^2},\ f = \frac{d^2U}{dydz},\ \&c.,\ l = \frac{dU}{dx},\ m = \frac{dU}{dy},\ n = \frac{dU}{dz}. \text{ (cf. Art. 102).]}$$

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* This is evidently a particular case of Ex. Art. 272.

275. *Through a point on a surface can be drawn* $(n+2)(n-3)$ *tangents which will also touch the surface elsewhere.*

In order that the line should touch at the point $x'y'z'w'$, we must, as before, have the coefficients of λ^n and $\lambda^{n-1}\mu = 0$; in consequence of which the equation we are considering becomes one of the $(n-2)^{\text{th}}$ degree, and if the line touch the surface a second time, this reduced equation must have equal roots. The condition that this should be the case involves the coefficients of that equation in the degree $2(n-3)$; one term, for instance, being $(\Delta^2 U'.\,U)^{n-3}$. By considering that term we see that this discriminant involves the coordinates $x'y'z'w'$ in the degree $(n-2)(n-3)$, and $xyzw$ in the degree $(n+2)(n-3)$. When therefore $x'y'z'w'$ is fixed, it denotes a surface which is met by the tangent plane in $(n+2)(n-3)$ right lines.

Thus, then, we have proved that at any point on a surface an infinity of tangent lines can be drawn: that these in general lie in a plane; that two of them pass through three consecutive points, and $(n+2)(n-3)$ of them touch the surface again.

276. Let us proceed next to consider the case of tangents drawn through a point not on the surface. Since we have in the preceding articles established relations which connect the coordinates of any point on a tangent with those of the point of contact, we can, by an interchange of accented and unaccented letters, express that it is the former point which is now supposed to be known, and the latter sought.

Thus, for example, making this interchange in the equation of Art. 273, we see that *the points of contact of all tangent lines (or of all tangent planes) which can be drawn through $x'y'z'w'$ lie on the first polar, which is of the degree* $(n-1)$: viz.

$$x'U_1 + y'U_2 + z'U_3 + w'U_4 = 0.$$

And since the points of contact lie also on the given surface, their locus is the curve of the degree $n(n-1)$, which is the intersection of the surface with the polar.

277. The assemblage of the tangent lines which can be drawn through $x'y'z'w'$ form a *cone*, the tangent planes to which are also tangent planes to the surface. The equation of this cone is found by forming the discriminant of the equation of the n^{th} degree in λ (Art. 272). For this discriminant expresses that the line joining the fixed point to $xyzw$ meets the surface in two coincident points; and therefore $xyzw$ may be a point on any tangent line through $x'y'z'w'$. *The discriminant is easily seen to be of the degree* $n\ (n-1)$, and it is otherwise evident that this must be *the degree of the tangent cone*. For its degree is the same as the number of lines in which it is met by any plane through the vertex. But such a plane meets the surface in a curve to which $n\ (n-1)$ tangents can be drawn through the fixed point, and these tangents are also the tangent lines which can be drawn to the surface through the given point.

[If the point be on the surface the tangent cone consists of the tangent plane and a proper cone of degree $n\ (n-1)-2$.

Ex. 1. The equation of any cubic surface (Cartesian coordinates), the origin being on the surface, $z=0$ being the tangent plane at the origin and x, y the inflexional tangents, is of form $cz + xy + zu_1 + u_3 = 0$, u_1 and u_3 being homogeneous in x, y, z. Prove that the tangent cone from the origin is

$$(xy + zu_1)^2 - 4czu_3 = 0.$$

Ex. 2. Prove (from the preceding or otherwise) that *the tangent cone from any point on a surface of any order touches the tangent plane along both inflexional tangents.* This is the reciprocal of the theorem of Art. 264. Observe that inflexional tangents reciprocate into inflexional tangents.

Ex. 3. Prove that the inflexional tangents at a point on a cubic surface are generators of the polar quadric of the point.

Ex. 4. In a point projection or "collineation" (Art. 144*c*) the projections of the asymptotic lines of the original surface are the asymptotic lines on the new surface.

Ex. 5. Find the equation of the tangent cone from $x'y'z'$ to the cylindroid $z(x^2+y^2) - 2hxy = 0$. Find the number of cuspidal edges of this cone and determine their position.]

278. *Through a point not on the surface can in general be drawn* $n\ (n-1)\ (n-2)$ *inflexional tangents.* We have seen (Art. 274) that the coordinates of any point on an inflexional tangent are connected with those of its point of contact by

the relation $U' = 0$, $\Delta U' = 0$, $\Delta^2 U' = 0$. If, then, we consider the $xyzw$ of any point on the tangent as known, its point of contact is determined as one of the intersections of the given surface U, which is of the n^{th} degree, with its first polar ΔU, which is of the $(n-1)^{\text{th}}$, and with the second polar $\Delta^2 U$, which is of the $(n-2)^{\text{th}}$. There are therefore $n(n-1)(n-2)$ such intersections. *If the point be on the surface, this number is diminished by six.* [For the diminution in the number can be seen to be the same for all surfaces of degree higher than *two*, and through a point on a cubic surface no lines can be drawn so as to be inflexional tangents at other points.]

279. *Through a point not on the surface can in general be drawn* $\frac{1}{2}n(n-1)(n-2)(n-3)$ *double tangents to it.* The points of contact of such lines are proved by Art. 275 to be the intersections of the given surface, of the first polar, and of the surface represented by the discriminant discussed in Art. 275, and which we there saw contained the coordinates of the point of contact in the degree $(n-2)(n-3)$. There are therefore $n(n-1)(n-2)(n-3)$ points of contact; and since there are two points of contact on each double tangent, there are half this number of double tangents. If the point be on the surface, the double tangents at the point (Art. 275) count each for two, and the number of lines through the point which touch the surface in two other points is

$$\tfrac{1}{2}n(n-1)(n-2)(n-3) - 2(n+2)(n-3)$$
$$= \tfrac{1}{2}(n^2+n+2)(n-3)(n-4).$$

Thus, then, we have completed the discussion of tangent lines which pass through a given point. We have shown that their points of contact lie on the intersection of the surface with one of the degree $n-1$, that their assemblage forms a cone of the degree $n(n-1)$, that $n(n-1)(n-2)$ of them are inflexional, and $\frac{1}{2}n(n-1)(n-2)(n-3)$ of them are double.

These latter double tangents are also plainly double edges of the tangent cone, since they belong to the cone in virtue of

each contact. Along such an edge can be drawn two tangent planes to the cone, namely the tangent planes to the surface at the two contacts.

The inflexional tangents, however, are also to be regarded as double tangents to the surface: since the line passing through three consecutive points is a double tangent in virtue of joining the first and second, and also of joining the second and third. The inflexional tangents are therefore double tangents whose points of contact coincide. They are therefore double edges of the tangent cone; but the two tangent planes along any such edge coincide. They are therefore cuspidal edges of the cone. We have proved, then, that *the tangent cone which is of the degree* $n(n-1)$ *has* $n(n-1)(n-2)$ *cuspidal edges, and* $\frac{1}{2}n(n-1)(n-2)(n-3)$ *double edges*; that is to say, any plane meets the cone in a section having such a number of cusps and such a number of double points.

280. Ex. 1. It is proved precisely as for plane curves (*Higher Plane Curves*, Art. 132), that if we take on each radius vector a length whose reciprocal is the n^{th} part of the sum of the reciprocals of the n radii vectores to the surface, then the locus of the extremity will be the polar plane of the point; that if the point be on the surface, the locus of the extremity of the mean between the reciprocals of the $n-1$ radii vectores will be the polar quadric, &c.

Ex. 2. By interchanging accented and unaccented letters in the equation of the polar plane, it is seen that the locus of the poles of all planes which pass through a given point is the first polar of that point.

Ex. 3. The locus of the pole of the plane which passes through two fixed points is hence seen to be a curve of the $(n-1)^2$ degree, namely, the intersection of the two first polars of those points. We see also that the first polar of every point on the line joining those two points must pass through the same curve.

Ex. 4. In like manner the first polars of any three points on a plane determine by their intersection $(n-1)^3$ points, any one of which is a pole of the plane, and through these points the first polars of every other point on the plane must pass.

281. From the theory of tangent lines drawn through a point we can in two ways derive *the degree of the reciprocal surface*. First, the number of points in which an arbitrary line meets the reciprocal is equal to the number of tangent planes which can be drawn to the given surface through a given line. Consider now any two points A and B on that

line, and let C be the point of contact of any tangent plane passing through AB. Then, since the line AC touches the surface, C lies on the first polar of A; and for the like reason it lies on the first polar of B. The points of contact, therefore, are the intersection of the given surface, which is of the n^{th} degree, with two polar surfaces, which are each of the degree $(n-1)$. The number of points of contact, and therefore *the degree of the reciprocal, is* $n(n-1)^2$.

282. Otherwise thus: let a tangent cone be drawn to the surface having the point A for its vertex; then since every tangent plane to the surface drawn through A touches this cone, the problem is, to find how many tangent planes to the cone can be drawn through any line AB; or if we cut the cone by any plane through B, the problem is to find how many tangent lines can be drawn through B to the section of the cone. But the class of a curve whose degree is $n(n-1)$, which has $n(n-1)(n-2)$ cusps, and $\frac{1}{2}n(n-1)(n-2)(n-3)$ double points, is

$$n(n-1)\{n(n-1)-1\}-3n(n-1)(n-2)$$
$$-n(n-1)(n-2)(n-3)=n(n-1)^2.$$

Generally the section of the reciprocal surface by any plane corresponds to the tangent cone to the original surface through any point.

Ex. Prove that *the degree* of the tangent cone to the reciprocal surface (as well as to the original surface) through any point is $n(n-1)$.

283. Returning to the condition that a line should touch a surface

$$xU_1'+yU_2'+zU_3'+wU_4'=0,$$

we see that *if all four differentials be made to vanish by the coordinates of any point, then every line through the point meets the surface in two coincident points, and the point is therefore a double point.* The condition that a given surface may have a double point is obtained by eliminating the variables between the four equations $U_1=0$, &c., and the function equated to zero is called the discriminant of the given surface (*Lessons on Higher Algebra*, Art. 105). The discriminant

being the result of elimination between four equations, each of the degree $n-1$, contains the coefficients of each in the degree $(n-1)^3$, and is therefore of the degree $4\ (n-1)^3$ in the coefficients of the original equation.

It is obvious from what has been said, that *when a surface has a double point, the first polar of every point passes through the double point.*

The surfaces represented by U_1, U_2, &c. may happen not merely to have points in common, but to have a whole curve common to all four surfaces. This curve will then be *a double curve on the surface* U, and every point of it will be a double point, such that the tangent cone resolves itself into a pair of planes. Now we saw (Art. 264) that the surface represented by the general Cartesian equation of the n^{th} degree will, in general, have an infinity of double tangent planes; the reciprocal surface therefore will, in general, have an infinity of double points, which will be ranged on a certain curve. *The existence then of these double curves is to be regarded among the "ordinary singularities" of surfaces.*

When the point $x'y'z'w'$ is a double point, U' and $\Delta U'$ vanish identically; and any line through the double point meets the surface in three consecutive points if it satisfies the equation $\Delta^2 U'=0$, which represents a cone of the second degree.

284. *The polar quadric of a parabolic point on a surface is a cone.*

[This follows from Art. 274, since the two inflexional tangents coincide at a parabolic point, and a cone is the only quadric that can have coincident generators at a point. The theorem may also be proved as follows:]

The polar quadric of the origin with regard to any surface

$$u_0 w^n + u_1 w^{n-1} + u_2 w^{n-2} + \&c. = 0$$

(where, as in Art. 272, we have introduced w so as to make the equation homogeneous) is found by differentiating $n-2$ times with respect to w. Dividing out by $(n-2)\ (n-3)\ \ldots\ 3$, and making $w=1$, the polar quadric is

$$n(n-1)\ u_0 + 2\ (n-1)\ u_1 + 2u_2 = 0.$$

Now the origin being a parabolic point, we have seen, Art. 266, that the equation is of the form

$$z + Cy^2 + 2Dzx + 2Ezy + Fz^2 + \&c.,$$

or, in other words, $u_0 = 0$, and u_2 is of the form $u_1v_1 + w_1^2$. The polar quadric then is

$$z(n - 1 + 2Dx + 2Ey + Fz) + Cy^2 = 0.$$

But any equation represents a cone when it is a homogeneous function of three quantities, each of the first degree. The equation just written therefore represents a cone whose vertex is the intersection of the three planes, z, $n - 1 + 2Dx + 2Ey + Fz$, and y. The two former planes are tangent planes to this cone, and y the plane of contact.

285. It follows from the last article, that *the locus of points whose polar quadrics are cones meets the given surface in its parabolic points*. This locus is found by writing down the discriminant of $\Delta^2 U' = 0$. If a, b, &c., denote the second differential coefficients $\frac{d^2U'}{dx'^2}$, $\frac{d^2U'}{dy'^2}$, &c., this discriminant will be a determinant formed with these coefficients, the developed result being (Art. 67)

$$abcd + 2afmn + 2bgnl + 2chlm + 2dfgh - bcl^2 - cam^2 - abn^2 - adf^2$$
$$- bdg^2 - cdh^2 + l^2f^2 + m^2g^2 + n^2h^2 - 2mngh - 2nlhf - 2lmfg = 0.$$

This denotes *a surface of the degree* 4 $(n - 2)$, which we shall call the *Hessian* of the given surface. In the same manner then, as the intersection of a plane curve with its Hessian determines the points of inflexion, so *the intersection of a surface with its Hessian determines a curve of the degree* $4n$ $(n - 2)$, *which is the locus of parabolic points* (see Art. 269).

286. It follows from what has been just proved that *through a given point can be drawn* $4n$ $(n - 1)$ $(n - 2)$ *stationary tangent planes* (see Art. 269). For since the tangent plane passes through a fixed point, its point of contact lies on the polar surface, whose degree is $n - 1$; and the intersection of this surface with the surface U, and the surface determined in the last article as the locus of points of contact of stationary tangent planes, determine $4n$ $(n - 1)$ $(n - 2)$ points.

Otherwise thus: the stationary tangent planes to the surface through any point are also stationary tangent planes to the tangent cone through that point, and if the cone be cut by any plane, these planes meet it in the tangents at the points of inflexion of the section. But the number of points of inflexion on a plane curve is determined by the formula (*Higher Plane Curves*, Art. 82)

$$\iota - \kappa = 3\ (\nu - \mu).$$

But in this case, Art. 282, we have $\nu = n\ (n-1)^2$, $\mu = n\ (n-1)$; therefore $\nu - \mu = n\,(n-1)\ (n-2)$, $\kappa = n\ (n-1)\,(n-2)$. Hence, as before, $\iota = 4n\ (n-1)\ (n-2)$.

The *number of double tangent planes* to the cone is determined by the formula

$$2(\tau - \delta) = (\nu - \mu)\ (\nu + \mu - 9),$$

where (Art. 282)

$$2\delta = n\ (n-1)\ (n-2)\ (n-3)\,;\ \nu + \mu - 9 = n^3 - n^2 - 9.$$

Hence $$2\tau = n\ (n-1)\ (n-2)\,(n^3 - n^2 + n - 12).$$

It follows then, that through any point can be drawn τ double tangent planes to the surface, where τ is the number just determined.

It will be proved hereafter, that the points of contact of double tangent planes lie on the intersection of the surface with one whose degree is

$$(n-2)\ (n^3 - n^2 + n - 12).$$

287. *If a right line lie altogether in a surface it will touch the Hessian and therefore the parabolic curve* (*Cambridge and Dublin Mathematical Journal*, Vol. IV., p. 255).

Let the equation of the surface be $x\phi + y\psi = 0$, and let us seek the result of making x and $y = 0$ in the equation of the Hessian, so as thus to find the points where the line meets that surface. Now, evidently, $\dfrac{d^2U}{dz^2}, \dfrac{d^2U}{dw^2}, \dfrac{d^2U}{dzdw}$ all contain x or y as a factor, and therefore vanish on this supposition. And if we make $c = 0$, $d = 0$, $n = 0$ in the equation of the Hessian, it becomes a perfect square $(fl - gm)^2$, showing that the right line touches the Hessian at every point where it meets it. If we make $x = 0$, $y = 0$ in

$fl - gm$, it reduces to $\dfrac{d\phi}{dz}\dfrac{d\psi}{dw} - \dfrac{d\phi}{dw}\dfrac{d\psi}{dz}$. It is evident that when the tangent plane touches all along any line, straight or curved, this line lies altogether in the Hessian, and not only so, but in the case of a straight line, it can be shown that the surface and the Hessian touch along this line.* The reader can verify this without difficulty, by using the form $x\phi + y^2\psi = 0$, the Hessian of which is of the same form $x\Phi + y^2\Psi = 0$.

Curvature of Surfaces.

288. We proceed next to investigate the curvature at any point on a surface of the various sections which can be made by planes passing through that point.

In the first place let it be premised that if the equation of a curve be $u_1 + u_2 + u_3 + \&c. = 0$, the radius of curvature at the origin is the same as for the conic $u_1 + u_2$. For it will be remembered that the ordinary expression for the radius of curvature includes only the coordinates of the point and the values of the first and second differential coefficients for that point. But if we differentiate the equation not more than twice, the terms got from differentiating u_3, u_4, &c. contain powers of x and y, and will therefore vanish for $x = 0$, $y = 0$. The values therefore of the differential coefficients for the origin are the same as if they were obtained from the equation $u_1 + u_2 = 0$.

It follows hence that the radius of curvature at the origin (the axes being rectangular) of $y + ax^2 + 2bxy + cy^2 + \&c. = 0$ is $\dfrac{1}{2a}$ (see *Conics*, Art. 241); or this value can easily be found directly from the ordinary expression for the radius of curvature (*Higher Plane Curves*, Art. 100).

289. Let now the equation of a surface referred to any tangent plane as plane of xy, and the corresponding normal as axis of z, be

$$z + Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + \&c. = 0,$$

* Cayley, "On Reciprocal Surfaces," *Phil. Trans.*, Vol. 159, 1869, see p. 208.

and let us investigate the *curvature of any normal section*, that is, of the section by any plane passing through the axis of z. Thus, to find the radius of curvature of the section by the plane xz, we have only to make $y=0$ in the equation, and we get a curve whose radius of curvature is half the reciprocal of A. In like manner the section by the plane yz has its radius of curvature = half the reciprocal of C. And in order to find the radius of curvature of any section whose plane makes an angle θ with the plane xz, we have only to turn the axes of x and y through an angle θ (by substituting $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y, *Conics*, Art. 9); and by then putting $y=0$ it appears, as before, that the radius of curvature is half the reciprocal of the new coefficient of x^2; that is to say,

$$\frac{1}{2R} = A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta.$$

290. The reader will not fail to observe that this expression for the radius of curvature of a normal section is identical in form with the expression for the square of the diameter of a central conic in terms of the angles which it makes with the axes of coordinates. Thus if ρ be the semi-diameter answering to an angle θ of the conic $Ax^2 + 2Bxy + Cy^2 = \frac{1}{2}$, we have $R = \rho^2$.

It may be seen, otherwise, that the radii of curvature are connected with their directions in the same manner as the squares of the diameters of a central conic. For we have seen that the radii of curvature depend only on the terms in u_1 and u_2. The radii of curvature therefore of all the sections of $u_1 + u_2 + u_3 +$ &c. are the same as those of the sections of the quadric $u_1 + u_2$; and it was proved (Art. 194) that these are all proportional to the squares of the diameters of the central section parallel to the tangent plane.

It is plain that the conic, the squares of whose radii are proportional to the radii of curvature, is similar to the indicatrix. [Thus the radius of curvature of any normal section is proportional, at any point, to the square of the

radius drawn to the indicatrix in the direction of the tangent line to the section, the indicatrix being supposed to lie in the tangent plane.]

[At a parabolic point $AC - B^2 = 0$ and

$$\frac{1}{2R} = (\sqrt{A} \cos \theta + \sqrt{C} \sin \theta)^2.$$

Thus R has always the same sign and becomes infinite along the direction of the inflexional tangent.]

291. We can now at once apply to the theory of these radii of curvature all the results that we have obtained for the diameters of central conics. Thus we know that the quantity $A \cos^2\theta + 2B \cos \theta \sin \theta + C \sin^2\theta$ admits of a maximum and minimum value; that the values of θ which correspond to the maximum and minimum are always real, and belong to directions at right angles to each other; and that those values of θ are given by the equation (see *Conics*, Art. 155)

$$B \cos^2\theta - (A - C) \cos \theta \sin \theta - B \sin^2\theta = 0.$$

Hence, *at any point on a surface there are among the normal sections, one for which the value of the radius of curvature is a maximum and one for which it is a minimum; the directions of these sections are at right angles to each other; and they are the directions of the axes of the indicatrix.* They plainly bisect the angles between the two inflexional tangents. We shall call these the *principal* sections, and the corresponding radii of curvature the *principal radii.*

If we turn round the axes of x and y so as to coincide with the directions of maximum and minimum curvature just determined, it is known that the quantity $Ax^2 + 2Bxy + Cy^2$ will take the form $A'x^2 + B'y^2$. Now the formula of Art. 289, when the coefficient of xy vanishes, gives the following expression for the half reciprocal of any radius of curvature $\frac{1}{2R} = A' \cos^2\theta + B' \sin^2\theta$. But evidently A' and B' are the values of this half reciprocal corresponding to $\theta = 0$, and $\theta = 90°$. Hence, *the radius of curvature of any normal section is expressed in terms of the two principal radii ρ and*

ρ', and of the angle which the direction of its plane makes with the principal planes, by the formula

$$\frac{1}{R} = \frac{\cos^2\theta}{\rho} + \frac{\sin^2\theta}{\rho'}.\text{*}$$

It is plain (as in *Conics*, Art. 157) that A' and B', or $\frac{1}{2\rho}, \frac{1}{2\rho'}$ are given by a quadratic equation, the sum of these quantities being $A + C$ and their product $AC - B^2$.

When $\rho = \rho'$, all the other radii of curvature are also $= \rho$. The form of the equation then is $z + A\ (x^2 + y^2) + \&c. = 0$, or the indicatrix is a circle. The origin is then an *umbilic*.

Ex. From the expressions in this article we deduce at once, as in the theory of central conics, that the *sum of the reciprocals of the radii of curvature of two normal sections at right angles to each other is constant*; and again, *if normal sections be made through a pair of conjugate tangents* (see Art. 268) *the sum of their radii of curvature is constant.*

292. It will be observed that the radius of curvature, being proportional to the *square* of the diameter of a central conic, does not become imaginary, but only changes sign, if the quantity $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ becomes negative. Now if radii of curvature directed on one side of the tangent plane are considered as positive, those turned the other way must be considered as negative; and the sign changes when the direction is changed in which the concavity of the curve is turned.

At an *elliptic* point on a surface; that is to say, when B^2 is less than AC, the sign of $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ remains the same for all values of θ; and therefore at such a point *the concavity of every section through it is turned in the same direction.*

At a *hyperbolic* point, that is to say, when B^2 is greater than AC, the radius of curvature twice changes sign, and *the concavity of some sections is turned in an opposite direction to that of others.* The surface, in fact, cuts the tangent plane in the neighbourhood of the point, and the inflexional

* This formula (with the inferences drawn from it) is due to Euler.

tangents mark the directions in which the surface crosses the tangent plane and divide the sections whose concavity is turned one way from those in which it is turned the other way.* And when we have chosen a hyperbola, the squares of whose diameters are proportional to one set of radii, then the other set of radii are proportional to the squares of the diameters of the conjugate hyperbola.

293. Having shown how to find the radius of curvature of any normal section, we shall next show how to express, in terms of this, the radius of curvature of any oblique section, inclined at an angle ϕ to the normal section, but meeting the tangent plane in the same line. Thus we have seen that the radius of curvature of the normal section made by the plane $y=0$ is half the reciprocal of A. Now let us turn the axes of y and z round in their plane through an angle ϕ (which is done by substituting $z\cos\phi - y\sin\phi$ for z, and $z\sin\phi + y\cos\phi$ for y). If we now make the new $y=0$, we shall get the equation (still to rectangular axes) of the section by a plane making an angle ϕ with the old plane $y=0$, but still passing through the old axis of x; and this equation will plainly be

$$0 = z\cos\phi + Ax^2 + 2(B\sin\phi + D\cos\phi)\,xz + (C\sin^2\phi + 2E\sin\phi\cos\phi + F\cos^2\phi)z^2 + \&c.$$

and by the same method as before the radius of curvature is found to be $\dfrac{\cos\phi}{2A}$, or is $= R\cos\phi$, where R is the radius of curvature of the corresponding normal section. This is MEUNIER'S THEOREM, that *the radius of curvature of an oblique section is equal to the projection on the plane of this section of the radius of curvature of a normal section passing through the same tangent line.* Thus we see that of all sections which can be made through any line drawn in the tangent plane, the normal section is that whose radius of

* The illustration of the summit of a mountain pass, or of a saddle, will enable the reader to conceive how a surface may in two directions sink below the tangent plane, and on the other sides rise above it; a mountain summit is an instance of an elliptic point. *Cf.* also the anchor-ring (Art. 266).

curvature is greatest; that is to say, the normal section is that which is least curved and which approaches most nearly to a straight line.

Meunier's theorem has been already proved in the case of a quadric (Art. 194), and we might therefore, if we had chosen, have dispensed with giving a new proof now; for we have seen that the radius of curvature of any section of $u_1 + u_2 + u_3 + \&c.$ is the same as that of the corresponding section of the quadric $u_1 + u_2$.

[Ex. Taking the axes of the indicatrix for axes of x and y, and the normal for axis of z, prove that the centres of curvature of plane sections at any point lie on the surface

$$(x^2 + y^2 + z^2)\left(\frac{x^2}{\rho'} + \frac{y^2}{\rho}\right) = z\,(x^2 + y^2).]$$

294. It was proved (Art. 203) that if two surfaces $u_1 + u_2 + \&c.$, $u_1 + v_2 + \&c.$ touch, their curve of intersection has a double point, the two tangents at which are the intersections of the plane u_1 with the cone $u_2 - v_2$. When the plane touches the cone, the surfaces have what we have called stationary contact. It is also proved, as at Art. 205, that *a sphere has stationary contact with a surface when the centre is on the normal and the radius equal to one of the principal radii of curvature.* In fact, the condition for stationary contact between

$$z + ax^2 + 2hxy + by^2 + \&c.,\quad z + a'x^2 + 2h'xy + b'y^2 + \&c.$$

is
$$(a - a')\,(b - b') = (h - h')^2,$$

which, when h and h' both vanish, implies either $a = a'$ or $b = b'$. The surface therefore $z + Ax^2 + By^2 + \&c.$ will have stationary contact with the sphere $2rz + x^2 + y^2 + z^2$ if $r = \dfrac{1}{2A}$ or $\dfrac{1}{2B}$; but these are the values of the principal radii.

295. The principles laid down in the last article enable us to find an *expression for the values of the principal radii at any point;* the axes of coordinates having any position.

If we transform the equation to any point $x'y'z'$ on the surface as origin, it becomes

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\frac{1}{1.2}\left(x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}\right)^2 U'+\&c.,$$

or, denoting the first differential coefficients by L, M, N, and the second by a, b, c, &c.,

$$2(Lx+My+Nz)+ax^2+by^2+cz^2+2fyz+2gzx+2hxy+\&c.=0.$$

The equation then of any sphere having the same tangent plane is, assuming the axes to be rectangular,

$$2(Lx+My+Nz)+\lambda(x^2+y^2+z^2)=0,$$

and this sphere will have stationary contact with the quadric if λ be determined so as to satisfy the condition that $Lx+My+Nz$ shall touch the cone

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy=0.$$

This condition is

$$\begin{vmatrix} a-\lambda & h, & g, & L \\ h, & b-\lambda & f, & M \\ g, & f, & c-\lambda & N \\ L, & M, & N, & \end{vmatrix}=0,$$

which expanded is

$$\{(b-\lambda)(c-\lambda)-f^2\}L^2+\{(c-\lambda)(a-\lambda)-g^2\}M^2$$
$$+\{(a-\lambda)(b-\lambda)-h^2\}N^2+2\{gh-(a-\lambda)f\}MN+$$
$$2\{hf-(b-\lambda)g\}NL+2\{fg-(c-\lambda)h\}LM=0,$$

or λ is given by the quadratic

$$(L^2+M^2+N^2)\lambda^2-\{(b+c)L^2+(c+a)M^2+(a+b)N^2$$
$$-2fMN-2gNL-2hLM\}\lambda$$
$$+(bc-f^2)L^2+(ca-g^2)M^2+(ab-h^2)N^2$$
$$+2(gh-af)MN+2(hf-bg)NL+2(fg-ch)LM=0.$$

Now if r be the radius of the sphere

$$\lambda(x^2+y^2+z^2)+2(Lx+My+Nz)=0,$$

we have $r^2=\dfrac{L^2+M^2+N^2}{\lambda^2}$. We therefore find the principal radii by substituting $\dfrac{\sqrt{(L^2+M^2+N^2)}}{r}$ for λ in the preceding quadratic.

The quadratic of this article might also have been found at once by Art. 102, which gives the axes of a section of the quadric

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

made parallel to the plane $Lx+My+Nz=0$. See Art. 290.

The absolute term in the equation for λ may be simplified by writing for L, M, N their values from the equations

$$(n-1)\,L = ax + hy + gz + lw, \text{ \&c.},$$

when the absolute term reduces to $-\dfrac{Hw^2}{(n-1)^2}$, where H is the Hessian, written at full length, Art. 285. We might have seen *a priori* that, for any point on the Hessian, the absolute term must vanish. For since the directions of the principal sections bisect the angles between the inflexional tangents; when the inflexional tangents coincide, one of the principal sections coincides with their common direction, and the radius of curvature of this section is infinite, since three consecutive points are on a right line. Hence one of the values of λ (which is the reciprocal of r) must vanish. [*If the surface is developable $H = 0$ at every point;* for the tangent plane meets the surface in two coincident right lines which are evidently the inflexional tangents; thus every point is parabolic.]

By equating to zero the coefficient of λ in the preceding quadratic, we obtain the equation of a surface of the degree $3n - 4$, which intersects the given surface in all the points where the principal radii are equal and opposite: that is to say, where the indicatrix is an equilateral hyperbola.

[Ex. 1. If the catenary $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ be rotated round its directrix (the axis of x) the principal radii at any point of the surface generated are equal and of opposite sign.

Surfaces having this property are called *minimal* surfaces. A minimal surface is a surface of minimum superficial area bounded by a given closed curve.

Ex. 2. If $y = x\phi(z)$ represent a minimal surface, prove that ϕ must be of the form $\tan(az + b)$.]

296. From the equations of the last article we can find the *radius of curvature of any normal section meeting the tangent plane in a line whose direction-angles are given.*

For the centre of curvature lies on the normal, and if we describe a sphere with this centre, and radius equal to the radius of curvature, it must touch the surface, and its equation is of the form

$$2(Lx + My + Nz) + \lambda(x^2 + y^2 + z^2) = 0.$$

The consecutive point on that section of the surface which we are considering satisfies this equation and also the equation $u_1 + u_2 = 0$, that is

$$2(Lx + My + Nz) + ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Subtracting, we find

$$\lambda = \frac{ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy}{x^2 + y^2 + z^2}.$$

And since this equation is homogeneous, we may write for x, y, z the direction-cosines of the line joining the consecutive point to the origin. As in the last article $\lambda = \dfrac{\surd(L^2 + M^2 + N^2)}{r}$.

Hence $r =$

$$\frac{\surd(L^2 + M^2 + N^2)}{a\cos^2\alpha + b\cos^2\beta + c\cos^2\gamma + 2f\cos\beta\cos\gamma + 2g\cos\gamma\cos\alpha + 2h\cos\alpha\cos\beta}.$$

The problem to find the maximum and minimum radius of curvature is, therefore, to make the quantity

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

a maximum or minimum, subject to the relations

$$Lx + My + Nz = 0, \quad x^2 + y^2 + z^2 = 1.$$

And thus we see, again, that this is exactly the same problem as that of finding the axes of the central section of a quadric by a plane $Lx + My + Nz$.

297. In like manner the problem to find the *directions of the principal sections at any point* is the same as to find the directions of the axes of the section by the plane $Lx + My + Nz$ of the quadric $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$.

Now given any diameter of a quadric, one section can be drawn through it having that diameter for an axis; the other axis being obviously the intersection of the plane perpendicular to the given diameter with the plane conjugate to it. Thus, if the central quadric be $U = 1$, and the given diameter pass through $x'y'z'$, the diameter perpendicular and conjugate is the intersection of the planes

$$xx' + yy' + zz' = 0, \quad x'U_1 + y'U_2 + z'U_3 = 0.$$

If the former diameter lie in a plane $Lx' + My' + Nz'$, the latter diameter traces out the cone which is represented by

the determinant obtained on eliminating $x'y'z'$ from the three preceding equations: viz.

$$(Mz - Ny)\ U_1 + (Nx - Lz)\ U_2 + (Ly - Mx)\ U_3 = 0.$$

And this cone must evidently meet the plane $Lx + My + Nz$ in the axes of the section by that plane. Thus, then, *the directions of the principal sections are determined as the intersection of the tangent plane $Lx + My + Nz$ with the cone*

$$(Mz - Ny)\ (ax + hy + gz) + (Nx - Lz)\ (hx + by + fz) + (Ly - Mx)\ (gx + fy + cz) = 0,$$

or

$$(Mg - Nh)\ x^2 + (Nh - Lf)\ y^2 + (Lf - Mg)z^2 + \{L\ (b - c) - Mh + Ng\}\ yz + \{Lh + M\ (c - a) - Nf\}\ zx + \{-Lg + Mf + N\ (a - b)\}\ xy = 0.$$

298. The methods used in Art. 295 enable us also easily to find the *conditions for an umbilic.** If the plane of xy be the tangent plane at an umbilic, the equation of the surface is of the form

$$z + A\ (x^2 + y^2) + 2Dxz + 2Eyz + Fz^2 + \&c. = 0;$$

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* It might be imagined that we could obtain a single condition for an umbilic by expressing that the quadratic (Art. 295) for the determination of the principal radii of curvature shall have equal roots. But, as at Art. 83, this quadratic, having its roots always real, is one of the class discussed in *Higher Algebra*, Art. 44, the discriminant of which can be expressed as a sum of squares. If we make these squares separately vanish, we obtain two conditions, which are more easily found as in the text.

In plane geometry, the problem of finding when $ax^2 + 2hxy + by^2 = 1$ represents a circle may be solved by taking the quadratic which gives the maximum or minimum values of $x^2 + y^2 = \rho$, viz. $(a\rho - 1)\ (b\rho - 1) - h^2\rho^2 = 0$, and forming the condition that the quadratic shall have equal roots, viz. $(a - b)^2 + 4h^2 = 0$. Now this single condition is not the condition that the curve shall be a circle, for either of the factors $a - b \pm 2hi$ separately equated to zero only expresses that the curve passes through one of the circular points at infinity. But if we have both factors simultaneously $= 0$, that is to say, if we have $a - b = 0$, $h = 0$, the curve passes through both circular points and is a circle. And the theory in regard to the umbilics is almost identical: the points on the surface for which the two radii of curvature are equal are points such that for each of them *one* of the inflexional tangents meets the imaginary circle at infinity; an umbilic is a point such that *both* the inflexional tangents meet the circle at infinity. The first-mentioned points form on the surface an imaginary locus having the umbilics for double points.

and if we subtract from it the equation of any touching sphere, viz.

$$z+\lambda\,(x^2+y^2+z^2)=0,$$

it is evidently possible so to choose λ (namely, by taking it $=A$) that all the terms in the remainder shall be divisible by z. We see, thus, that if u_1+u_2+ &c. represent the surface, and $u_1+\lambda v_2$ any touching sphere, it is possible, when the origin is an umbilic, so to choose λ that $u_2-\lambda v_2$ may contain u_1 as a factor. We see, then, by transformation of coordinates as in Art. 295, that any point $x'y'z'$ will be an umbilic if it is possible so to choose λ that

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy$$

may contain as a factor $Lx+My+Nz$. If so, the other factor must be

$$\frac{a-\lambda}{L}x+\frac{b-\lambda}{M}y+\frac{c-\lambda}{N}z.$$

Multiplying out and comparing coefficients of yz, zx, xy, we get the conditions

$$(b-\lambda)\frac{N}{M}+(c-\lambda)\frac{M}{N}=2f,\ (c-\lambda)\frac{L}{N}+(a-\lambda)\frac{N}{L}=2g,$$

$$(a-\lambda)\frac{M}{L}+(b-\lambda)\frac{L}{M}=2h.$$

Eliminating λ between these equations, we obtain *for an umbilic the two conditions*

$$\frac{bN^2+cM^2-2fMN}{N^2+M^2}=\frac{cL^2+aN^2-2gLN}{L^2+N^2}=\frac{aM^2+bL^2-2hLM}{M^2+L^2}.$$

Since there are only two conditions to be satisfied, *a surface of the n^{th} degree has in general a determinate number of umbilics;* for the two conditions, each of which represents a surface, combined with the equation of the given surface, determine a certain number of points. It may happen, however, that the surfaces represented by the two conditions intersect in a curve which lies (either wholly or in part) on the given surface. In such a case there will be on the given surface a line, every point of which will be an umbilic. Such a line is called *a line of spherical curvature.*

[Ex. 1. Prove that the umbilics of $xyz = a^3$ are at the four points

$$x = \pm a,\ y = \pm a,\ z = \pm a,$$

and that the radius of curvature at an umbilic is equal to the distance of the umbilic from the origin.

Prove that the surface is synclastic, and find the principal radii at any point.

Ex. 2. Prove that the sphere of radius r and centre at the origin touches the surface

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n + \left(\frac{z}{c}\right)^n = 1,$$

at the umbilics, where

$$r^{\frac{2n}{n-2}} = a^{\frac{2n}{n-2}} + b^{\frac{2n}{n-2}} + c^{\frac{2n}{n-2}}$$

and that the radius of curvature at the umbilics is $\dfrac{r}{n-1}$.

Assuming abc are positive, if n is even the surface is everywhere synclastic. If n is odd the coordinate planes separate the synclastic from the anticlastic portions and cut the surface in parabolic curves.]

299. Before applying the conditions of the last article, the form in which we have written them requires that the following considerations should be attended to.

These equations appear to be satisfied by making $L = 0$, $a = \dfrac{bN^2 + cM^2 - 2fNM}{N^2 + M^2}$; whence we might conclude that the surface $L = 0$ must always pass through umbilics on the given surface. Now it is easy to see geometrically that this is not the case, for L (or U_1) is the polar of the point yzw with respect to the surface, so that if L necessarily passed through umbilics it would follow by transformation of coordinates that the first polar of *every* point passes through umbilics. On referring to the last article, however, it will be seen that the investigation tacitly assumes that none of the quantities L, M, N vanish; for if any of them did vanish, some of the equations which we have used would contain infinite terms. Supposing then L to vanish, we must examine directly the condition that $My + Nz$ may be a factor in

$$(a-\lambda)x^2 + (b-\lambda)y^2 + (c-\lambda)z^2 + 2fyz + 2gzx + 2hxy.$$

We must evidently have $\lambda = a$, and it is then easily seen that we must, as before, have $a = \dfrac{bN^2 + cM^2 - 2fMN}{N^2 + M^2}$, while in addition, since the terms $2gzx + 2hxy$ must be divisible by

$My + Nz$, we must have $Mg = Nh$. Combining then with the two conditions here found, $L = 0$, and the equation of the surface, there are four conditions which, except in special cases, cannot be satisfied by the coordinates of any points.

If we clear of fractions the conditions given in the last article, it will be found that they each contain either L, M, or N as a factor. And what we have proved in this article is that these factors may be suppressed as irrelevant to the question of umbilics.

Again, it can be shown that, introducing homogeneous coordinates as in Art. 295, the numerators of the above fractions multiplied by $(n-1)^2$, are respectively

$$n\,(n-1)\,(bc-f^2)\,U - (Dx^2 + Aw^2 - 2Lxw),$$
$$n\,(n-1)\,(ca-g^2)\,U - (Dy^2 + Bw^2 - 2Myw),$$
$$n\,(n-1)\,(ab-h^2)\,U - (Dz^2 + Cw^2 - 2Nzw),$$

where A, B, C, D, L, M, N are the functions of a, b, c, &c. defined in Art. 67. Hence our equations are satisfied for $U = 0$ by $w = 0$, $D = 0$, but these are the points of inflexion of the intersection of U with the plane at infinity, which are also irrelevant to the question of umbilics.

From what has been said we can infer the number of umbilics which a surface of the n^{th} degree will in general possess. We have seen that the umbilics are determined as the intersection of the given surface with a curve whose equations are of the form $\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}$. Now if A, B, C be of the degree l, and A', B', C' of the degree m, then $AB' - BA'$, $AC' - CA'$ are each of the degree $l + m$, and intersect in a curve of the degree $(l+m)^2$. But the intersection of these two surfaces includes the curve AA' of the degree lm which does not lie on the surface $BC' - CB'$. The degree therefore of the curve common to the three surfaces is $l^2 + lm + m^2$. In the present case $l = 3n - 4$, $m = 2n - 2$, and the degree of the curve would seem to be $19n^2 - 46n + 28$. But we have seen that the system we are discussing includes three curves such as

$$L,\ a\,(M^2 + N^2) - (bN^2 + cM^2 - 2fMN)$$

which do not pass through umbilics. Subtracting therefore from the number just found $3\,(n-1)\,(3n-4)$, we see that the umbilics are determined as the intersection of the given surface with a curve of the degree $(10n^2 - 25n + 16)$, but from the number of points thus found we must subtract $3n\,(n-2)$ for the inflexions on the intersection of the given surface with the plane at infinity. Thus *the number of umbilics* is $n\,(10n^2 - 28n + 22)$. (Voss, *Math. Annalen*, IX. 1876). In particular, when $n = 2$, then the number is twelve, viz. there are four umbilics in each of the principal planes.

We now proceed to draw some other inferences from what was proved (Art. 294); namely, that the two principal spheres have stationary contact with the surface.

300. *When two surfaces have stationary contact, they touch in two consecutive points.*

The equations of the two surfaces being

$$z+ax^2+2hxy+by^2+\&c.=0,\ z+a'x^2+2h'xy+b'y^2+\&c.,$$

the tangent planes at a consecutive point are (Art. 268)

$$z+2\ (ax'+hy')\ x+2\ (hx'+by')\ y=0,$$
$$z+2\ (a'x'+h'y')\ x+2\ (h'x'+b'y')\ y=0.$$

That these may be identical, we must have

$$ax'+hy'=a'x'+h'y',\ hx'+by'=h'x'+b'y',$$

and eliminating $x' : y'$ between these equations, we have

$$(a-a')\ (b-b')=(h-h')^2,$$

which is the condition for stationary contact. (Art. 204.)

The sphere, therefore, whose radius is equal to one of the principal radii, touches the surface in two consecutive points; or two consecutive normals to the surface are also normals to the sphere, and consequently intersect in its centre. Now we know that in plane curves the centre of the circle of curvature may be regarded as the intersection of two consecutive normals to the curve. In surfaces the normal at any point will not meet the normal at a consecutive point taken arbitrarily. But we see here that *if the consecutive point be taken in the direction of either of the principal sections, the two consecutive normals will intersect, and their common length will be the corresponding principal radius.* On account of the importance of this theorem we give a direct investigation of it.

301. *To find in what cases the normal at any point on a surface is intersected by a consecutive normal.* Take the tangent plane for the plane of xy, and let the equation of the surface be

$$z+Ax^2+2Bxy+Cy^2+2Dxz+2Eyz+Fz^2+\&c.=0.$$

Then we have seen (Art. 268) that the equation of a consecutive tangent plane is

$$z+2(Ax'+By')\ x+2\ (Bx'+Cy')\ y=0,$$

and a perpendicular to this through the point $x'y'$ will be

$$\frac{x-x'}{Ax'+By'}=\frac{y-y'}{Bx'+Cy'}=2z.$$

This will meet the axis of z (which was the original normal) if

$$\frac{x'}{Ax'+By'}=\frac{y'}{Bx'+Cy'}.$$

The direction therefore of a consecutive point whose normal meets the given normal is determined by the equation

$$Bx'^2+(C-A)x'y'-By'^2=0.$$

But this is the same equation (Art. 291) which determines the directions of maximum and minimum curvature. *At any point on a surface therefore there are two directions, at right angles to each other, such that the normal at a consecutive point taken on either intersects the original normal. And these directions are those of the two principal sections at the point.* Taking for greater simplicity the directions of the principal sections as axes of coordinates; that is to say, making $B=0$ in the preceding equations, the equations of a consecutive normal become $\frac{x-x'}{Ax'}=\frac{y-y'}{Cy'}=2z$, whence it is easy to see that the normals corresponding to the two points, for one of which $y'=0$, and for the other $x'=0$, intersect the axis of z at distances determined respectively by $2Az+1=0$, $2Cz+1=0$. *The intercepts therefore on a normal by the two consecutive ones which intersect it are equal to the principal radii.**

* Bertrand, in his theory of the curvature of surfaces, calculates the angle made by the consecutive normal with the plane containing the original normal and the consecutive point $x'y'$. Supposing still the directions of the principal sections to be axes of coordinates, the direction-cosines of the consecutive normal are proportional to $2Ax'$, $2Cy'$, 1, while those of a tangent line perpendicular to the radius vector are proportional to $-y'$, x', 0. Hence the cosine of the angle between these two lines, or the sine of the angle which the consecutive normal makes with the normal section, is proportional to $2(C-A)x'y'$; or, if α be the angle which the direction of the consecutive point makes with one of the principal tangents, is proportional to $(C-A)\sin 2\alpha$. When $\alpha=0$, or $=90°$, this angle vanishes, and the consecutive normal is in the plane of the original normal.

We may also arrive at the same conclusions by seeking the locus of points on a surface, the normals at which meet a fixed normal which we take for axis of z. Making $x=0$, $y=0$ in the equation of any other normal, we see that the point where it meets the surface must satisfy the condition $U_2x=U_1y$. The curve where this surface meets the given surface has the extremity of the given normal for a double point, the two tangents at which are the two principal tangents to the surface at that point. (See Ex. 11, Art. 121.)

The special case where the fixed normal is one at an umbilic deserves notice. The equation of the surface being of the form $z+A\ (x^2+y^2)+\&c.=0$, the lowest terms in the equation $xU_2=yU_1$, when we make $z=0$, will be of the third degree, and the umbilic is a triple point on the curve locus. Thus while every normal immediately consecutive to the normal at the umbilic meets the latter normal, there are *three directions along any of which the next following normal will also meet the normal at the umbilic.*

Hamilton has pointed out (*Elements of Quaternions*, Art. 411) how this is verified in the case of a quadric. He has proved that the two imaginary generators (see Art. 139) through any umbilic are lines of curvature, the third line of curvature through the umbilic being the principal section in which it lies. In fact, for a point on a principal section, the cone (Ex. 11) breaks up into two planes. The normal therefore at such a point only meets the normals at the points of the principal section, and at the points of another plane section. For the umbilic the latter plane is a tangent plane and the section reduces to the imaginary generators. The normals along either lie in the same imaginary plane. At every point in either generator, distinct from the umbilic, the two directions of curvature coincide with the line, which is perpendicular to itself (*Conics*, p. 351). There is, however, some speciality as regards the theory of the umbilics of a quadric.

[Ex. 1. If s is the length of an arc PQ of a curve drawn on a surface, and d the shortest distance between the normals to the surface at P and Q, *the limiting value of* $\frac{d}{s}$ *as* Q *moves towards coincidence with* P *is equal to the cosine of the angle between the tangent to the curve at* P *and the conjugate tangent line to the surface at* P (Art. 267). Hence these limiting values for different curves through P may be represented on the indicatrix.

Ex. 2. If C is the centre of curvature of a principal section at P, all the normals in the neighbourhood of P intersect the line through C perpendicular to the section (Sturm).

Ex. 3. Show that the points where the surface

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=1$$

meets the line

$$\frac{x}{a^3}=\frac{y}{b^3}=\frac{z}{c^3}$$

are umbilics, and that the directions of the three lines of curvature through an umbilic are given by

$$\frac{dy}{b^3}=\frac{dz}{c^3},\ \frac{dz}{c^3}=\frac{dx}{a^3},\ \frac{dx}{a^3}=\frac{dy}{b^3}.]$$

302. *A line of curvature** on a surface is a line traced on it, such that the normals at any two consecutive points of it intersect. Thus, starting with any point M on a surface, we may go on to either of the two consecutive points N, N', whose normals were proved to intersect the normal at M. The normal at N, again, is intersected by the consecutive normals at two points, P, P', the element NP being a continuation of the element MN while the element NP' is approximately perpendicular to it. In like manner we might pass from the point P to another consecutive point Q, and so have a line of curvature $MNPQ$. But we might evidently have pursued the same process had we started in the direction MN'. Hence *at any point M on a surface can be drawn two lines of curvature; these cut at right angles and are touched by the two "principal tangents" at M.* A line of curvature will ordinarily not be a plane curve, and even in the special case where it is plane it need not coincide with a principal normal section at M, though it must touch such a section. For the principal section must be normal to the surface, and the line of curvature may be oblique.

A very good illustration of lines of curvature is afforded by the case of a *surface generated by the revolution of any plane curve* round an axis in its plane. At any point P of such a surface one line of curvature is the plane section passing through P and through the axis, or, in other words, is the generating curve which passes through P. For, all the normals to this curve are also normals to the surface, and, being in one plane, they intersect. The corresponding principal radius at P is evidently the radius of curvature of the

* The whole theory of lines of curvature, umbilics, &c., is due to Monge. See his *Application de l'Analyse à la Géométrie*, p. 124, Liouville's edition.

plane section at the same point. The other line of curvature at P is the circle which is the section made by a plane drawn through P perpendicular to the axis of the surface; for the normals at all the points of this section evidently intersect the axis of the surface at the same point, and therefore intersect each other. The intercept on the normal between P and the axis is plainly the second principal radius of the surface.

The generating curve which passes through P is a principal section of the surface, since it contains the normal and touches a line of curvature; but the section perpendicular to the axis is, in general, not a principal section because it does not contain the normal at P. The second principal section at that point would be the plane section drawn through the normal at P and through the tangent to the circle described by P.

The example chosen serves also to illustrate Meunier's theorem; for the radius of the circle described by P (which, as we have seen, is an oblique section of the surface) is the projection on that plane of the intercept on the normal between P and the axis, and we have just proved that this intercept is the radius of curvature of the corresponding normal section.

303. It was proved (Art. 297) that the direction-cosines of the tangent line to a principal section fulfil the relation

$$\begin{aligned}&(M\cos\gamma - N\cos\beta)\,(a\cos\alpha + h\cos\beta + g\cos\gamma)\\ &+(N\cos\alpha - L\cos\gamma)\,(h\cos\alpha + b\cos\beta + f\cos\gamma)\\ &+(L\cos\beta - M\cos\alpha)\,(g\cos\alpha + f\cos\beta + c\cos\gamma) = 0.\end{aligned}$$

Now the tangent line to a principal section is also the tangent to the line of curvature; while, if ds be the element of the arc of any curve, the projections of that element upon the three axes being dx, dy, dz, it is evident that the cosines of the angles which ds makes with the axes are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$. The *differential equation of the lines of curvature* is therefore got by writing dx, dy, dz for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in the preceding formula.

This equation may also be found directly as follows (see Gregory's *Solid Geometry*, p. 256): Let α, β, γ be the coordinates of a point common to two consecutive normals. Then, if xyz be the point where the first normal meets the surface, by the equations of the normal we have $\frac{\alpha - x}{L} = \frac{\beta - y}{M} = \frac{\gamma - z}{N}$; or, if we call the common value of these fractions θ, we have

$$\alpha = x + L\theta,\ \beta = y + M\theta,\ \gamma = z + N\theta.$$

But if the second normal meet the surface in a point

$$x + dx,\ y + dy,\ z + dz,$$

then, expressing that $\alpha\beta\gamma$ satisfies the equations of the second normal, we get the same results as if we differentiate the preceding equations, considering $\alpha\beta\gamma$ as constant, or

$$dx + Ld\theta + \theta dL = 0,\quad dy + Md\theta + \theta dM = 0,\quad dz + Nd\theta + \theta dN = 0,$$

from which equations eliminating θ, $d\theta$, we have the same determinant as in Art. 297, viz.

$$\begin{vmatrix} dx, & dy, & dz \\ L, & M, & N \\ dL, & dM, & dN \end{vmatrix} = 0.$$

Of course

$$dL = adx + hdy + gdz,\ dM = hdx + bdy + fdz,$$
$$dN = gdx + fdy + cdz.$$

[Evidently we may substitute for L, M, N, in this differential equation, the direction-cosines of the normal (say λ, μ, ν) or we may substitute $p\lambda$, $p\mu$, $p\nu$, where p is a function of x, y, z.]

Ex. 1. *To find the differential equation of the lines of curvature of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here we have

$$L = \frac{x}{a^2},\ M = \frac{y}{b^2},\ N = \frac{z}{c^2},\quad dL = \frac{dx}{a^2},\quad dM = \frac{dy}{b^2},\quad dN = \frac{dz}{c^2}.$$

Substituting these values in the preceding equation it becomes, when expanded,

$$(b^2 - c^2)\, x\, dy\, dz + (c^2 - a^2)\, y\, dz\, dx + (a^2 - b^2)\, z\, dx\, dy = 0.$$

Knowing, as we do, that the lines of curvature are the intersections of the ellipsoid with a system of confocal quadrics (Art. 196), it would be easy to assume for the integral of this equation $Ax^2 + By^2 + Cz^2 = 0$, and to determine the constants by actual substitution. If we assume nothing as to the form of the integral we can eliminate z and dz by the help of the equation of the surface, and so get a differential equation in two variables which is the equation of the projection of the lines of curvature on the plane xy. Thus, in the present case, multiplying by $\frac{z}{c^2}$ and reducing by the equation of the ellipsoid and its differential, we have

$$\{(b^2 - c^2)\, xdy + (c^2 - a^2)\, ydx\} \left\{\frac{xdx}{a^2} + \frac{ydy}{b^2}\right\} = (a^2 - b^2) \left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\} dx\, dy,$$

or writing $$\frac{a^2 (b^2 - c^2)}{b^2 (a^2 - c^2)} = A,\ \frac{a^2 (a^2 - b^2)}{a^2 - c^2} = B,$$

$$Axy\left(\frac{dy}{dx}\right)^2 + (x^2 - Ay^2 - B)\frac{dy}{dx} - xy = 0,$$

the integral of which is, with C an arbitrary constant,

$$\frac{x^2}{B} - \frac{y^2}{BC} = \frac{1}{AC+1},$$

or *the lines of curvature are projected on the principal plane into a series of conics* whose axes a', b' are connected by the relation

$$\frac{a'^2(a^2-c^2)}{a^2(a^2-b^2)} + \frac{b'^2(b^2-c^2)}{b^2(b^2-a^2)} = 1.$$

It is not difficult to see that this coincides with the account given of the lines of curvature in Art. 196. [We find the projections by eliminating z between the equation of the ellipsoid and the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.]$$

[Ex. 2. The orthogonal projections, on the plane $x = 0$, of the lines of curvature of an ellipsoid consist of a system of coaxal ellipses and of portions of a system of hyperbolas having the same axes, the real axis coinciding with the axis of z. The internal and external portions of the lines joining the projections of the umbilics are limiting cases of the ellipses and hyperbolas respectively. The section by $x = 0$ is a limiting case of an ellipse, and the axis of z of a hyperbola. The projections on the plane $z = 0$ are of like nature.*

The projections on $y = 0$ are portions of two systems of ellipses.

Ex. 3. Two symmetrically placed lines of curvature can be found, such that their orthogonal projections on the plane $x = 0$, are circles, and the radius of both circles is $\frac{bc}{a}$.

Find the confocal passing through these lines of curvature.

Similarly two lines of curvature can be found projecting into *portions* of a circle on the plane $y = 0$; but there is no like circle for the z plane.

Ex. 4. Investigate the nature of projections on principal planes of the lines of curvature of the hyperboloids and paraboloids.

Ex. 5. The lines of curvature of a cone are the generators and the intersections of the cone with concentric spheres.]

304. The theorem that confocal quadrics intersect in lines of curvature is a particular case of a theorem due to Dupin, which we shall state as follows: *If three surfaces intersect at right angles, and if each pair also intersect at right angles at their next consecutive common point, then the directions of the intersections are the directions of the lines of curvature on each.*

Take the point common to all three surfaces as origin, and the three rectangular tangent planes as coordinate planes; then the equations of the surfaces are of the form

* See illustration, Fig. 1, Chap. V.

$$\begin{aligned} x+ay^2 &+2byz &+cz^2 &+2dzx &+ \&c. = 0, \\ y+a'z^2 &+2b'zx &+c'x^2 &+2d'xy &+ \&c. = 0, \\ z+a''x^2 &+2b''xy &+c''y^2 &+ \&c. &= 0. \end{aligned}$$

At a consecutive point common to the first and second surfaces, we must have $x=0$, $y=0$, $z=z'$, where z' is very small. The consecutive tangent planes are

$$(1+2dz')x + \quad 2bz'y \quad + \; 2cz'z = 0,$$
$$2b'z'x + (1+2d'z')y + 2a'z'z = 0.$$

Forming the condition that these should be at right angles and only attending to the terms where z' is of the first degree, we have $b+b'=0$.

In like manner, in order that the other pairs of surfaces may cut at right angles at a consecutive point, we must have $b'+b''=0$, $b''+b=0$, and the three equations cannot be fulfilled unless we have b, b', b'' each separately $=0$; in which case the form of the equation shows (Art. 301) that the axes are the directions of the lines of curvature on each. Hence follows the theorem in the form given by Dupin; * namely, that *if there be three systems of surfaces, such that every surface of one system is cut at right angles by all the surfaces of the other two systems, then the intersection of two surfaces belonging to different systems is a line of curvature on each.* For, at each point of it, it is, by hypothesis, possible to draw a third surface cutting both at right angles.

[Dupin's theorem may be deduced symmetrically, without choosing special axes, from the differential equation for the lines of curvature (Art. 303).

Let U, V, W be the three surfaces and let $l_1 = \frac{dU}{dx}$, $m_1 = \frac{dU}{dy}$, &c., $l_2 = \frac{dV}{dx}$, &c., $l_3 = \frac{dW}{dx}$, &c.

Let δx, δy, δz be the coordinate projections of an element of the curve U, V at a point common to U, V, W. Then

$$l_1\delta x + m_1\delta y + n_1\delta z = 0,$$
$$l_2\delta x + m_2\delta y + n_2\delta z = 0.$$

Also

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0,$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0,$$

therefore $\delta x : \delta y : \delta z :: l_3 : m_3 : n_3$.

* *Développements de Géométrie*, 1813, p. 330. The demonstration here given is by Prof. W. Thomson: see Gregory's *Solid Geometry*, p. 263. *Cambridge Mathematical Journal*, Vol. IV., p. 62. See also the proof by R. L Ellis, Gregory's *Examples*, p. 215.

We have therefore to prove

$$\begin{vmatrix} l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \\ \delta l_1 & \delta m_1 & \delta n_1 \end{vmatrix} = 0.$$

Which expresses the condition (Art. 303) that δx, δy, δz may be elements of a line of curvature on $U = 0$.

But
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0,$$
$$l_2 l_1 + m_2 m_1 + n_2 n_1 = 0,$$
therefore the determinant will vanish if
$$l_2 \delta l_1 + m_2 \delta m_1 + n_2 \delta n_1 = 0.$$
Using $\delta l_1 = a_1 \delta x + h_1 \delta y + g_1 \delta z$, &c., and substituting for $\delta x : \delta y : \delta z$ as before, this requires
$$a_1 l_2 l_3 + b_1 m_2 m_3 + c_1 n_2 n_3 + f_1(m_2 n_3 + m_3 n_2) + \&c. = 0.$$

Let this equation be $P_1 = 0$. It expresses the condition that, if U, V, W cut mutually at right angles, then UV and UW are lines of curvature on U.

We can show that this condition is satisfied if each pair cut at right angles at their next consecutive point. If this is true of U and V then
$$d(l_1 l_2 + m_1 m_2 + n_1 n_2) = 0$$
or $d\Delta_3 = 0$ along UV. Hence
$$\frac{d\Delta_3}{dx}\delta x + \frac{d\Delta_3}{dy}\delta y + \frac{d\Delta_3}{dz}\delta z = 0,$$
substituting for $\delta x : \delta y : \delta z$ as before and using
$$\frac{d\Delta_3}{dx} = a_1 l_2 + a_2 l_1 + h_2 m_1 + h_1 m_2 + \&c.,$$
we get $P_1 + P_2 = 0$.

Similarly $P_2 + P_3 = 0$ since V, W intersect at right angles at their next point, and $P_3 + P_1 = 0$ since the same is true of W, U. Hence $P_1 = P_2 = P_3 = 0$.

A closely connected theorem is the following: *If two surfaces cut at a constant angle, and if their intersection is a line of curvature on one, it is also a line of curvature on the other* (Joachimsthal).

Let $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ be the direction cosines of the normals to U and V. We have to prove that the determinant

$$\begin{vmatrix} \delta x & \delta y & \delta z \\ \alpha_1 & \beta_1 & \gamma_1 \\ \delta\alpha_1 & \delta\beta_1 & \delta\gamma_1 \end{vmatrix} = 0$$

if the curve is a line of curvature on V.

We have as before
$$\delta x : \delta y : \delta z :: (\beta_1\gamma_2) : (\gamma_1\alpha_2) : (\alpha_1\beta_2)$$
substituting in the above determinant it may be expressed as the product of the two arrays

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} \quad \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \delta\alpha_1 & \delta\beta_1 & \delta\gamma_1 \end{vmatrix}$$
$$= \begin{vmatrix} \alpha_1^2 + \beta_1^2 + \gamma_1^2 & \alpha_1\delta\alpha_1 + \beta_1\delta\beta_1 + \gamma_1\delta\gamma_1 \\ \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 & \alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 \end{vmatrix}$$

But $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$ therefore $\alpha_1\delta\alpha_1 + \beta_1\delta\beta_1 + \gamma_1\delta\gamma_1 = 0$.

Hence *the condition that the curve UV may be a line of curvature on U is* $\alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 = 0$, and similarly the condition that it may be a line of curvature on V is $\alpha_1\delta\alpha_2 + \beta_1\delta\beta_2 + \gamma_1\delta\gamma_2 = 0$. But if one of these conditions is satisfied so is the other since $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2$ is constant.]

305. A line of curvature is, by definition, such that the normals to the surface at two consecutive points of it intersect each other. If, then, we consider *the surface generated by all the normals along a line of curvature, this will be a developable surface* (Note, Art. 112), since two consecutive generating lines intersect. The developable generated by the normals along a line of curvature manifestly *cuts the given surface at right angles.*

The locus of points where two consecutive generators of a developable intersect is a curve whose properties will be more fully explained in the next chapter, it is called the *cuspidal edge* or *edge of regression* of that developable. Each generator is a tangent to this curve, for it joins two consecutive points of the curve; namely, the points where the generator in question is met by the preceding and by the succeeding generator see (Art. 123).

Consider now the normal at any point M of a surface; through that point can be drawn two lines of curvature $MNPQ$, &c., $MN'P'Q'$, &c.: let the normals at the points M, N, P, Q, &c., intersect in C, D, E, &c., and those at M, N', P', Q', in C', D', E'; then it is evident that the curve CDE, &c., is the cuspidal edge of the developable generated by the normals along the first line of curvature, while $C'D'E'$ is the cuspidal edge of the developable generated by the normals along the second. The normal at M, as has just been explained, touches these curves at the points C, C', which are the two centres of curvature corresponding to the point M.

What has been proved may be stated as follows: *The cuspidal edge of the developable generated by the normals along a line of curvature is the locus of one of the systems of centres of curvature corresponding to all the points of that line.*

306. The assemblage of the centres of curvature C, C' answering to all the points of a surface is a surface of two sheets, called the *surface of centres* (see Art. 198). The curve CDE lies on one sheet while $C'D'E'$ lies on the other sheet. *Every normal to the given surface touches both sheets of the surface of centres:* for it has been proved that the normal at M touches the two curves CDE, $C'D'E'$, and every tangent line to a curve traced on a surface is also a tangent to the surface.

Now if from a point, not on a surface, be drawn two consecutive tangent lines to the surface, the plane of those lines is manifestly a tangent plane to the surface; for it is a tangent plane to the cone which is drawn from the point touching the surface. But if two consecutive tangent lines intersect on the surface, it cannot be inferred that their plane touches the surface. For if we cut the surface by any plane whatever any two consecutive tangents to the curve of section (which, of course, are also tangent lines to the surface) intersect on the curve, and yet the plane of these lines is supposed not to touch the surface.

Consider now the two consecutive normals at the points M, N, these are both tangents to both sheets of the surface of centres. And since the point C in which they intersect is on the first sheet but not necessarily on the second, the plane of the two normals is the tangent plane to the second sheet of the surface of centres.

The plane of the normals at the points M, N' is the tangent plane to the other sheet of the surface of centres. But because the two lines of curvature through M are at right angles to each other, it follows that these two planes are at right angles to each other. Hence *the tangent planes to the surface of centres at the two points C, C', where any normal meets it, cut each other at right angles.*

307. It is manifest that for every umbilic on the given surface the two sheets of the surface of centres have a point common; or, in other words, the surface of centres has a

double point; and if the original surface have a line of spherical curvature, the surface of centres will have a double line. The two sheets will cut at right angles everywhere along this double line.

This, however, is not the only case where the surface of centres has a double line. A double point on that surface arises not only when the two centres which belong to the same normal coincide, but also when two different normals intersect, and the point of intersection is a centre of curvature for each. It was shown, Arts. 298-9, that a surface of the n^{th} degree possesses ordinarily a definite number of umbilics, and, therefore, in general not a line of spherical curvature. Hence a double line of the first kind is not among the ordinary singularities of the surface of centres. But that surface will in general have a double line of the second kind. Through any point $n(n^2 - n + 1)$ normals can be drawn to a surface: *every* point on the surface of centres is a centre of curvature for one of these normals, each point of a certain locus on the surface will be a centre of curvature for two normals, and there will even be a definite number of points each a centre of curvature for three normals.*

308. It is convenient to define here a *geodesic line* on a surface, and to establish the fundamental property of such a line; namely, that its osculating plane (see Art. 123) at any point contains the normal to the surface. A geodesic line is

* The possibility of double lines of the second kind was overlooked by Monge and by succeeding geometers; and, oddly enough, first came to be recognized in consequence of Kummer's having had a model made of the surface of centres of an ellipsoid (see *Monatsberichte* of the Berlin Academy, 1862). Instead of finding the sheets, as he expected, to meet only in the points corresponding to the umbilics, he found that they intersected in a curve, and that they did not cut at right angles along this line. Of course when the existence of the double line was known to be a fact its mathematical theory was evident. Clebsch had, on purely mathematical grounds, independently arrived at the same conclusion in an elaborate paper on the normals to an ellipsoid, of equal date with Kummer's paper, though of later publication. A discussion of the surface of centres of an ellipsoid, founded on Clebsch's paper, will be given in chapter XIV.

the form assumed by a strained thread lying on a surface and joining any two points on the surface. It is plain that *the geodesic is ordinarily the shortest line on the surface by which the two points can be joined,* since, by pulling at the ends of the thread, we must shorten it as much as the interposition of the surface will permit. Now the resultant of the tensions along two consecutive elements of the curve, formed by the thread, lies in the plane of those elements, and since it must be destroyed by the resistance of the surface, it is normal to the surface; hence, *the plane of two consecutive elements (the osculating plane) of the geodesic contains the normal to the surface.**

The same thing may also be proved geometrically. In the first place, if two points A, C in different planes be connected by joining each to a point B in the intersection of the two planes, the sum of AB and BC will be less than the sum of any other joining lines AB', $B'C$, if AB and BC make equal angles with TT', the intersection of the planes. For if one plane be made to revolve about TT' until it coincide with the other, AB and BC become one right line, since the angle TBA is supposed to be equal to $T'BC$; and the right line AC is the shortest by which the points A and C can be joined.

It follows, that if AB and BC be consecutive elements of a curve traced on a surface, that curve will be the shortest line connecting A and C when AB and BC make equal

* I have followed Monge in giving this proof, the mechanical principles which it involves being so elementary that it seems pedantic to object to their introduction. For the benefit of those who prefer a purely geometrical proof, one or two are added in the text. For readers familiar with the theory of maxima and minima it is scarcely necessary to add that *a geodesic need not be the absolutely shortest line by which two points on the surface may be joined.* Thus, if we consider two points on a sphere joined by a great circle, the remaining portion of that great circle, exceeding 180°, is a geodesic, though not the shortest line connecting the points. [And through two points on a cylinder an infinite number of geodesics can be drawn, viz. the helices through the points. A geodesic, however, is the shortest line joining two of its points if their distance apart, measured along the geodesic, is small enough.

A geodesic is also the path described by a particle moving on a smooth surface, and subject to no forces except the reaction of the surface.]

angles with BT, the intersection of the tangent planes at A and C.

We see, then, that AB (or its production) and BC are consecutive edges of a right cone having BT for its axis. Now the plane containing two consecutive edges is a tangent plane to the cone; and since every tangent plane to a right cone is perpendicular to the plane containing the axis and the line of contact, it follows that the plane ABC (the osculating plane to the geodesic) is perpendicular to the plane AB, BT, which is the tangent plane at A. The theorem of this article is thus established.

Bertrand has remarked (*Liouville*, t. XIII., p. 73, cited by Cayley, *Quarterly Journal*, Vol. I., p. 186) that this fundamental property of geodesics follows at once from Meunier's theorem (see Art. 293). For it is evident, that for an indefinitely small arc, the chord of which is given, the excess in length over the chord is so much the less as the radius of curvature is greater. The shortest arc, therefore, joining two indefinitely near points A, B, on a surface is that which has the greatest radius of curvature, and we have seen that this is the normal section.

309. Returning now to the surface of centres, I say that the curve CDE (Art. 306), which is *the locus of points of intersection of consecutive normals along a line of curvature, is a geodesic on the sheet of the surface of centres on which it lies*. For we saw (Art. 306) that the plane of two consecutive normals to the surface (that is to say, the plane of two consecutive tangents to this curve) is the tangent plane to the second sheet of the surface of centres and is perpendicular to the tangent plane at C to that sheet of the surface of centres on which C lies. Since, then, the osculating plane of the curve CDE is always normal to the surface of centres, the curve is a geodesic on that surface.

310. We have given the equations connected with lines of curvature on the supposition that the equation of the sur-

face is presented, as it ordinarily is, in the form $\phi(x, y, z)=0$. As it is convenient, however, that the reader should be able to find here the formulæ which have been commonly employed, we conclude this chapter by deriving the principal equations in the form given by Monge and by many subsequent writers, viz. when the equation of the surface is in the form $z=\phi(x, y)$. We use the ordinary notations

$$dz=pdx+qdy,\ dp=rdx+sdy,\ dq=sdx+tdy.$$

We might derive the results in this form from those found already; for since $U=\phi(x, y)-z=0$, we have

$$\frac{dU}{dx}=p,\ \frac{dU}{dy}=q,\ \frac{dU}{dz}=-1,$$

with corresponding expressions for their second differential coefficients. We shall, however, repeat the investigations for this form as they are usually given.

The equation of a tangent plane is

$$z-z'=p(x-x')+q(y-y'),$$

and the equations of the normal are

$$(x-x')+p(z-z')=0,\ y-y'+q(z-z')=0.$$

If then $\alpha\beta\gamma$ be any point on the normal, and xyz the point where it meets the surface, we have

$$(\alpha-x)+p(\gamma-z)=0,\ (\beta-y)+q(\gamma-z)=0.$$

And if $\alpha\beta\gamma$ also satisfy the equations of a second normal, the differentials of these equations must vanish, or

$$dx+pdz=(\gamma-z)\,dp,\ dy+qdz=(\gamma-z)\,dq\ ;$$

whence, eliminating $(\gamma-z)$, we have the equation of condition

$$(dx+pdz)\,dq=(dy+qdz)\,dp,$$

[which is a special form of the equation of Art. 303, when L, M, N are replaced by $p, q-1$].

Putting in for dz, dp, dq their values already given, and arranging, we have

$$\frac{dy^2}{dx^2}\{(1+q^2)s-pqt\}+\frac{dy}{dx}\{(1+q^2)r-(1+p^2)t\}-\{(1+p^2)s-pqr\}=0.$$

This equation determines *the projections on the plane of xy of the lines of curvature* at any point, there being two through each point.

311. From the equations of the preceding article we can also find the *lengths of the principal radii.* The equations

$$dx + pdz = (\gamma - z)\ dp,\ dy + qdz = (\gamma - z)\ dq,$$

when transformed as above become

$$\{1 + p^2 - (\gamma - z)r\}\ dx + \{pq - (\gamma - z)s\}\ dy = 0,$$
$$\{1 + q^2 - (\gamma - z)t\}\ dy + \{pq - (\gamma - z)s\}\ dx = 0,$$

whence eliminating $dx : dy$, we have

$$(\gamma - z)^2(rt - s^2) - (\gamma - z)\ \{\ (1 + q^2)r - 2pqs + (1 + p^2)t\}$$
$$+ (1 + p^2 + q^2) = 0.$$

Now $\gamma - z$ is the projection of the radius of curvature on the axis of z; and the cosine of the angle the normal makes with that axis being $\dfrac{1}{\sqrt{(1 + p^2 + q^2)}}$ we have,

$$R = (\gamma - z)\ \sqrt{(1 + p^2 + q^2)}.$$

Eliminating then $\gamma - z$ by the help of the last equation, R is given by the equation

$$R^2(rt - s^2) - R\{\ (1 + q^2)r - 2pqs + (1 + p^2)t\}\ \sqrt{(1 + p^2 + q^2)}$$
$$+ (1 + p^2 + q^2)^2 = 0.$$

[Ex. Prove that the sphere is the only *real* surface whose principal radii are everywhere equal.]

312. From the preceding results can be deduced Joachimsthal's theorem (see *Crelle*, Vol. XXX., p. 347) that *if a line of curvature be a plane curve, its plane makes a constant angle with the tangent plane to the surface at any of the points where it meets it.* Let the plane be $z = 0$, then the equation of Art. 310

$$(dx + pdz)dq = (dy + qdz)dp$$

becomes $dx\,dq = dy\,dp$. But we have also $pdx + qdy = 0$, consequently $pdp + qdq = 0$ and $p^2 + q^2 = \text{constant}$. But $p^2 + q^2$ is the square of the tangent of the angle which the tangent plane makes with the plane xy, since $\cos\gamma = \dfrac{1}{\sqrt{(1 + p^2 + q^2)}}$.

Otherwise thus (see *Liouville*, Vol. XI., p. 87): Let MM', $M'M''$ be two consecutive and equal elements of a line of curvature, then the two consecutive normals are two perpendiculars to these lines passing through their middle points

I, I', and C the point of meeting of the normals is equidistant from the lines MM', $M'M''$. But if from C we let fall a perpendicular CO on the plane $MM'M''$, O will be also equidistant from the same elements; and therefore the angle $CIO = CI'O$. It is proved then that the inclination of the normal to the plane of the line of curvature remains unchanged as we pass from point to point of that line.

More generally let the line of curvature not be plane. Then, as before, the tangent planes through MM' and through $M'M''$ make equal angles with the plane $MM'M''$. And evidently the angle which the second tangent plane makes with a second osculating plane $M'M''M'''$ differs from the angle which it makes with the first by the angle between the two osculating planes. Thus we have Lancret's theorem, that *along a line of curvature the variation in the angle between the tangent plane to the surface and the osculating plane to the curve is equal to the angle between the two consecutive osculating planes.**

For example, *if a line of curvature be a geodesic it must be plane.* For then the angle between the tangent plane and osculating plane does not vary, being always right; therefore the osculating plane itself does not vary.

313. Finally, to obtain the *radius of curvature of any normal section.* Since the centre of curvature $\alpha\beta\gamma$ lies on the normal, we have

$$(\alpha - x) + p(\gamma - z) = 0,\ (\beta - y) + q(\gamma - z) = 0.$$

Further, we have

$$(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2 = R^2.$$

And since this relation holds for three consecutive points of the section which is osculated by the circle we are considering, we have

$$(\alpha - x)dx + (\beta - y)dy + (\gamma - z)dz = 0,$$
$$(\alpha - x)d^2x + (\beta - y)d^2y + (\gamma - z)d^2z = dx^2 + dy^2 + dz^2.$$

* We shall see in the next chapter that this is equivalent to the theorem that *the geodesic torsion of a line of curvature is zero.*

Combining this last with the preceding equations, we have

$$\frac{\alpha - x}{p} = \frac{\beta - y}{q} = -\frac{\gamma - z}{1} = \frac{R}{\sqrt{1+p^2+q^2}} = \frac{dx^2+dy^2+dz^2}{pd^2x+qd^2y-d^2z}.$$

But differentiating the equation $dz = pdx + qdy$, we have

$$d^2z - pd^2x - qd^2y = rdx^2 + 2sdxdy + tdy^2,$$

whence $R = \pm\sqrt{(1+p^2+q^2)}\dfrac{dx^2+dy^2+(pdx+qdy)^2}{rdx^2+2sdxdy+tdy^2}.$

The radius of curvature, therefore, of a normal section the projection of whose tangent line on the plane of xy is parallel to $y = mx$ is

$$\pm\sqrt{(1+p^2+q^2)}\frac{(1+p^2)+2pqm+(1+q^2)m^2}{r+2sm+tm^2}.$$

The conditions for an umbilic are got by expressing that this value is independent of m (since every normal section has the same radius of curvature), and are

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t}.$$

[Ex. 1. The umbilics of the surface of revolution $z = f(x^2+y^2)$ are the intersections of the surface with the circular cylinders (coaxal with the surface) determined by the equation

$$\frac{d^2f}{du^2} = 2\left(\frac{df}{du}\right)^3, \text{ where } u = x^2 + y^2.$$

The radius of curvature at an umbilic is

$$\frac{\sqrt{1+4u\left(\dfrac{df}{du}\right)^2}}{2\dfrac{df}{du}}.$$

Ex. 2. The differential equation of the projection of the asymptotic lines on the plane of xy is

$$rdx^2 + 2sdxdy + tdy^2 = 0.$$

Ex. 3. The surface formed by the revolution of a parabola round its directrix is everywhere anticlastic, and one principal radius is double the other (cf. conclusion of Art. 302).]

CHAPTER XII.

CURVES AND DEVELOPABLES.

Section I. Projective Properties.

314. It was proved (Art. 22) that two equations represent a curve in space. Thus the equations $U=0$, $V=0$ represent the curve of intersection of the surfaces U, V.

*The degree of a curve in space is measured by the number of points in which it is met by any plane.** Thus, if U, V be of the m^{th} and n^{th} degrees respectively, the surfaces which they represent are met by any plane in curves of the same degrees, which intersect in mn points. The curve UV is therefore of the mn^{th} degree.

By eliminating the variables alternately between the two given equations, we obtain three equations

$$\phi(y, z)=0, \ \psi(z, x)=0, \ \chi(x, y)=0,$$

which are the equations of the projections of the curve on the three coordinate planes. Any one of the equations taken separately represents the cylinder whose edges are parallel to one of the axes, and which passes through the curve (Art. 25). The theory of elimination shows that the equation $\phi(y, z)=0$ obtained by eliminating x between the given equations is of the mn^{th} degree. And it is also geometrically evident that any cone or cylinder standing on a curve of the r^{th} degree is of the r^{th} degree. For if we draw any plane through the vertex of the cone [or parallel to the generators of the cylinder] this plane meets the cone in r lines; namely, the lines joining the vertex to the r points where the plane meets the curve.

* [When the degree of a curve is spoken of, it is assumed that the curve is algebraic (p. 344).]

315. Now, conversely, if we are given any curve in space and desire to represent it by equations, we need only take the three plane curves which are the projections of the curve on the three coordinate planes; then any two of the equations $\phi(y, z) = 0$, $\psi(z, x) = 0$, $\chi(x, y) = 0$ will represent the given curve. But ordinarily these will not form the simplest system of equations by which the curve can be represented. For if r be the degree of the curve, these cylinders being each of the r^{th} degree, any two intersect in a curve of degree r^2; that is to say, not merely in the curve we are considering but in an extraneous curve of the degree $r^2 - r$. And if we wish not only to obtain a system of equations satisfied by the points of the given curve, but also to exclude all extraneous points, we must preserve the system of three projections; for the projection on the third plane of the extraneous curve in which the first two cylinders intersect will be different from the projection of the given curve.

It *may* be possible by combining the equations of the three projections to arrive at two equations $U = 0$, $V = 0$, which shall be satisfied for the points of the given curve, and for no other. But it is not generally true that *every* curve in space is the complete intersection of two surfaces. To take the simplest example, consider two quadrics having a right line common, as, for example, two cones having a common edge. The intersection of these surfaces, which is in general of the fourth degree, must consist of the common right line, and of a curve of the third degree. Now since the only factors of 3 are 1 and 3, a curve of the third degree cannot be the complete intersection of two surfaces unless it be a plane curve; but the curve we are considering cannot be a plane curve,* for if so any arbitrary line in its plane would meet it in three points, but such a line could not meet either quadric in more points

* Curves in space which are not plane curves have commonly been called "curves of double curvature". In what follows, I use the word "curve" to denote a curve in space, which ordinarily is not a plane curve, and I add the adjective "twisted" when I want to state expressly that the curve is not a plane curve.

than two, and therefore could not pass through three points of their curve of intersection.

316. The question thus arises *how to represent in general an algebraic curve in space by equations* (see p. 344). Several answers may be given.

(A) Generalizing the method at the beginning of the last article, we may consider a set of surfaces $U=0$, $V=0$, $W=0$, &c. (where U, V, W, . . . are rational and integral functions of the coordinates), all passing through the given curve. This being so, if M, N, P, &c., are also rational and integral functions of the coordinates, then $MU+NV+PW+\ldots=0$ is a surface passing through the curve. If any one of the original equations can be thus represented by means of the other equations, *e.g.* if we have identically $U=NV+PW+\ldots$, we reject this equation; and if we have through the curve any surface whatever $T=0$ which is not thus representable (viz. if T is not of the form $T=MU+NV+PW+\ldots$), then we join on the equation $T=0$ to the original system; and so on: if, as may happen, the adjunction of any new equation renders a former equation superfluous, such former equation is to be rejected. We thus arrive at a *complete* system of surfaces passing through the given curve, viz such a system is $U=0$, $V=0$, $W=0$, . . . where these functions are not connected by any such equation as $U=NV+PW+\ldots$, and where every other surface which passes through the curve is expressible in the form $MU+NV+PW+\ldots=0$. It is not easy to prove, but it may safely be assumed, that for a curve of any given order whatever, the number of equations in such a complete system is finite. And we have thus the representation of a curve in space by means of a complete system of surfaces passing through it.

(B) Taking as vertex an arbitrary point, the cone passing through a given curve of the order m is, as we have seen, of the order m; and it is such that each generating line meets the curve once only. Hence we can on each generating line of a cone of the order m determine a single point in such-

wise that the locus of these points is a curve of the order m. It would at first sight appear that we might thus determine the curve as the intersection of the cone by a surface of the order n, having at the vertex of the cone an $(n-1)$-ple point; for then each generating line of the cone meets the surface in the vertex counting $(n-1)$ times, and in one other point. But the curve of intersection is not then in general a curve of the order m, but is a curve of the order mn having a singular point at the vertex. To cause this curve to be of the order m, the surface of the order n with the $(n-1)$-ple point must be particularized; such a surface has through the multiple point $n(n-1)$ right lines; and if any one or more of these lines are on the cone, the complete intersection of the cone and surface will include as part of itself such line or lines, and there will be a residual curve of an order less than mn, and which may reduce itself to m; viz. the complete intersection of the cone and surface will then consist of $m(n-1)$ lines through the vertex (or rather of lines counting this number of times), and of a residual curve of the order m. *The analytical representation of the curve (using quadriplanar coordinates) is by means of two equations, the cone* $(x, y, z)^m = 0$, *and the* monoid $(x, y, z)^n + w(x, y, z)^{n-1} = 0$ *particularized as above.**

(*C*) *The coordinates of any point of a curve in space may be given as functions of a single parameter* θ. They cannot in general be thus expressed as *rational* functions of θ, for this would be a restriction on the generality of the curve in space (the curve would in fact be *unicursal*); but if we imagine two parameters θ, ϕ connected by an algebraic equation, then the coordinates of the point of the curve in space may be taken to be rational functions of θ, ϕ. Or, what is the same thing, writing $\frac{\xi}{\zeta}$ and $\frac{\eta}{\zeta}$ instead of θ, ϕ, we have between ξ, η, ζ an equation $(\xi, \eta, \zeta)^m = 0$, and then (using for the curve in space quadriplanar coordinates) x, y, z, w

* See Cayley, *Comptes rendus*, t. LIV. (1862), pp. 55, 396, 672.

proportional to rational and integral functions $(\xi, \eta, \zeta)^n$; we thus determine the curve in space, by expressing the coordinates of any point thereof rationally in terms of the coordinates of a point of the plane curve $(\xi, \eta, \zeta)^m = 0$.*

(*D*) *A curve in space will be determined if we determine all the right lines which meet it; viz. if we establish between the six coordinates of a right line the relation which expresses that the line meets the curve.* Such relation is expressed by a single equation $(p, q, r, s, t, u)^m = 0$ between the coordinates of a right line.

[This equation represents a *complex* of the m^{th} order (Art. 80 *f*); all the lines of the complex passing through any point are the generators of the cone with its vertex at the point and standing on the curve. An example of this mode of representing a curve is given in Art. 217. The order of the complex is the same as the degree of the cone from any point, and is therefore the same as the degree of the curve, since the number of generators of the cone lying in a plane through the vertex is equal to the number of points in which the plane meets the curve.]

But the difficulty is that, not every such equation, but only an equation of the proper form, expresses that the right line meets a determinate curve in space. For example, the general linear relation $(p, q, r, s, t, u)^1 = 0$ is not the equation of any line in space; the particular form

$$ps' + qt' + ru' + sp' + tq' + ur' = 0,$$

where (p', q', r', s', t', u') are constants such that

$$p's' + q't' + r'u' = 0$$

is the equation of a right line, viz. of the line the six coordinates of which are (p', q', r', s', t', u'); in fact, the equation obviously expresses that the line (p, q, r, s, t, u) meets this line.

[In order to express the curve of intersection of two surfaces U and V by a complex, we employ the method used Ex. 9, Art. 121, to find the cone joining the curve to $x'y'z'w'$. The result of eliminating λ can be expressed as a homogeneous relation of degree mn between the six coordinates $yz' - y'z$, &c., m and n being the degrees of U and V.

* [Non-algebraic (transcendental) curves (p. 344) may also be defined by parameters.]

A *surface* may also be represented by a complex, viz. that formed by its tangent lines. We express the discriminant of the equation

$$U + \lambda\delta U + \frac{\lambda^2}{1.2}\delta^2 U + \&c.,$$

as a homogeneous equation in $yz' - y'z$, &c.

But a complex in general represents neither a curve nor a surface.

Ex. To find the equation of the complex of lines passing through the intersection of the surface with the plane $L \equiv ax + \beta y + \gamma z + \delta w = 0$.

We find the point where the line joining P_1 to P_2 meets the plane $L = 0$ by substituting $x_1 + \lambda x_2$, &c., for x, y, z, w, and the coordinates of the point are found to be $\beta r - \gamma q + \delta s$, $\gamma p + \delta t - ar$, $\delta u + aq - \beta p$, $-as - \beta t - \gamma u$. Substituting these for x, y, z, w in $f(x, y, z, w) = 0$, we get the required complex, which represents the plane section.

(E) A fifth method of representing a curve in space is by means of its two *intrinsic equations*; this will be explained in Art. 368 (b). This method is independent of particular axes, and it enables us only to determine the *intrinsic* properties of the curve, that is, its form and dimensions. It is not specially applicable to algebraic curves.]

317. If a curve be either the complete or partial intersection of two surfaces U, V, the *tangent to the curve at any point* is evidently the intersection of the tangent planes to the two surfaces, and is represented by the equations

$$xU'_1 + yU'_2 + zU'_3 + wU'_4 = 0,$$
$$xV'_1 + yV'_2 + zV'_3 + wV'_4 = 0.$$

When we use rectangular axes, the direction-cosines of the tangent are plainly proportional to $MN' - M'N$, $NL' - N'L$, $LM' - L'M$, where L, M, &c., are the first differential coefficients.

An exceptional case arises when the two surfaces touch, in which case the point of contact is a double point on their curve of intersection. All this has been explained before (see Art. 203). As a particular case of the above, the projection of the tangent line to any curve is the tangent to its projection; and when the curve is given as the intersection of the two cylinders $y = \phi(z)$, $x = \psi(z)$, the equations of the tangent are

$$y - y' = \frac{d\phi}{dz}(z - z'), \; x - x' = \frac{d\psi}{dz}(z - z').$$

This may be otherwise expressed as follows: Consider any element of the curve ds; it is projected on the axes of coordinates into dx, dy, dz. The direction-cosines of this element are therefore $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, and *the equations of the tangent are*

$$\frac{x - x'}{\frac{dx}{ds}} = \frac{y - y'}{\frac{dy}{ds}} = \frac{z - z'}{\frac{dz}{ds}}.$$

[If the curve is expressed by a single parameter θ, the equation of the tangent is got by substituting $d\theta$ for ds in this equation, since $\frac{dx}{d\theta} = \frac{dx}{ds} \cdot \frac{ds}{d\theta}$.]

Since the sum of the squares of the three cosines is equal to unity, we have $ds^2 = dx^2 + dy^2 + dz^2$.

We shall postpone to another section the theory of normals, radii of curvature, and in short everything which involves the consideration of angles, and in this section we shall only consider what may be called the projective properties of curves [Art. 144c].

318. The theory of curves is in a great measure identical with that of *developables*, on which account it is necessary to enter more fully into the latter theory. In fact it was proved (Art. 123) that the reciprocal of a series of points forming a curve is a series of planes enveloping a developable. We there showed that the points of a curve regarded as a system of points give rise to a *system of lines*, viz. the tangents to the curve; and that they also give rise to a *system of planes*, viz. those containing every three consecutive points of the system, these planes being the *osculating planes of the curve*. *The assemblage of the lines of the system forms a developable surface whose equation can be found when the equation of the curve is given.* For, the two equations of the tangent line to the curve involve the three coordinates x', y', z',

which being connected by two relations are reducible to a single parameter; and by the elimination of this parameter from the two equations, we obtain the equation of the surface. Or, in other words, we must eliminate $x'y'z'$ between the two equations of the tangent and the two equations of the curve.

We have said (Art. 123) that the surface generated by the tangents is a *developable*, since every two consecutive positions of the generating line intersect each other. The name given to this kind of surface is derived from the property that it can be unfolded into a plane without crumpling or tearing. Thus, imagine any series of lines Aa, Bb, Cc, Dd, &c. (which for the moment we take at finite distances from each other) and such that each intersects the consecutive in the points a, b, c, &c.; and suppose a surface to be made up of the faces AaB, BbC, CcD, &c., then it is evident that such a surface could be developed into a plane by turning the face AaB round aB as a hinge until it formed a continuation of BbC; by turning the two, which we had thus made into one face, round cC until they formed a continuation of the next face, and so on. In the limit when the lines Aa, Bb, &c., are indefinitely near, the assemblage of plane elements forms a developable which, as just explained, can be unfolded into one plane.

The reader will find no difficulty in conceiving this from the examples of developables with which he is most familiar, viz. a cone or a cylinder. There is no difficulty in folding a sheet of paper into the form of either surface and in unfolding it again into a plane. But it will easily be seen to be impossible to fold a sheet of paper into the form of a sphere (which is not a developable surface); or, conversely, if we cut a sphere in two it is impossible to make the portions of the surface lie smooth in one plane.

But in order to exhibit better the form of a developable surface, as also its cuspidal curve afterwards referred to, take two sheets of paper, and cutting out from these two equal circular annuli (*e.g.* let the radii of the two circles be 3 inches and $4\frac{1}{2}$ inches), and placing these one upon the other, gum

them together along the inside edge by means of short strips of muslin or thin paper; we have thus a double annulus, which, so long as it remains complete, can only be bent in the same way as if it were single; but cutting through the double annulus along a radius, and taking hold of the two extremities, the whole can be opened out into *two sheets* of a developable surface, of which the inner circle, bending into a curve of double curvature, is the cuspidal curve or edge of regression.*

It is to be added, that if we draw on each of the two sheets the tangents to the inner circle, and consider each tangent as formed of two halves separated by the point of contact, then when the paper is bent into a developable surface as above, a set of half-tangents on the one sheet will unite with a set of half-tangents on the other sheet to form the generating lines on the developable surface; while the remaining two sets of half-tangents will unite to form on the developable surface a set of curves of double curvature, each touching a generating line at a point of the cuspidal curve, in the manner that a plane curve touches its tangent at a point of inflexion.

[Ex. Find the equation of the developable generated by tangent lines to the curve

$$x = \theta,\ y = \theta^2,\ z = \theta^3.]$$

319. The plane AaB containing two consecutive generating lines is evidently, in the limit, a tangent plane to the developable. It is obvious that we might consider the surface as generated by the motion of the plane AaB according to some assigned law, the envelope of this plane in all its positions being the developable. Now if we consider the developable generated by the tangent lines of a curve in space, the equations of the tangent at any point $x'y'z'$ are plainly functions of those coordinates, and the equation of the plane containing any tangent and the next consecutive (in other words, the equation of the osculating plane at any point $x'y'z'$) is also a function of these coordinates. But since $x'y'z'$ are connected

* Thompson and Tait's *Natural Philosophy*, Art. 149. Cayley mentions that he believes the construction is due to Blackburn.

by two relations, namely, the equations of the curve, we can eliminate any two of them, and so arrive at this result, that *a developable is the envelope of a plane whose equation contains a single variable parameter.*

[For example to *express in one parameter the plane coordinates of the developable formed by tangent lines to the curve*

$$x' = f_1(\theta),\ y' = f_2(\theta),\ z' = f_3(\theta).$$

The problem is the same as to find the tangent plane to the developable at the point θ on the curve. Let $\lambda x + \mu y + \nu z + \rho$ be this tangent plane. Then $\lambda f_1(\theta) + \mu f_2(\theta) + \nu f_3(\theta) + \rho = 0$. Again, the plane passes through the tangent line at θ, namely,

$$\frac{x - f_1(\theta)}{f_1'(\theta)} = \frac{y - f_2(\theta)}{f_2'(\theta)} = \frac{z - f_3(\theta)}{f_3'(\theta)}.$$

Therefore $\lambda f_1'(\theta) + \mu f_2'(\theta) + \nu f_3'(\theta) = 0$ and it passes through the consecutive tangent line, therefore $\lambda f_1''(\theta) + \mu f_2''(\theta) + \nu f_3''(\theta) = 0$.

Eliminating λ, μ, ν, ρ the equation of the tangent plane is

$$\begin{vmatrix} x & y & z & 1 \\ f_1 & f_2 & f_3 & 1 \\ f_1' & f_2' & f_3' & 0 \\ f_1'' & f_2'' & f_3'' & 0 \end{vmatrix} = 0.$$

We find the equation of the developable by eliminating θ from the equations of the tangent line.]

To make this principle better understood we shall point out an important difference between the cases when a plane curve is considered as the envelope of a moveable line, and when a surface in general is considered as the envelope of a moveable plane.

320. The equation of the tangent to a plane curve is a function of the coordinates of the point of contact ; and these two coordinates being connected by the equation of the curve, we can either eliminate one of them, or else express both in terms of a third variable so as to obtain the equation of the tangent as a function of a single variable parameter. The converse problem, to obtain the envelope of a right line whose equation includes a variable parameter has been discussed, *Higher Plane Curves*, Art. 86. Let the equation of any tangent line be $u = 0$, where u is of the first degree in x and y, and the constants are functions of a parameter t. Then

the line answering to the value of the parameter $t+h$ is $u+\frac{du}{dt}\frac{h}{1}+\frac{d^2u}{dt^2}\frac{h^2}{1.2}+$ &c.; and the point of intersection of these two lines is given by the equations $u=0$, $\frac{du}{dt}+\frac{h}{1.2}\frac{d^2u}{dt^2}+$ &c. $=0$. And, in the limit, the point of intersection of a line with the next consecutive (or, in other words, the point of contact of any line with its envelope) is given by the equations $u=0$, $\frac{du}{dt}=0$. If from these two equations we eliminate t we obtain the locus of the points of intersection of each line of the system with the next consecutive; that is to say, the equation of the envelope of all these lines. It is easy to prove that the result of this elimination represents a curve to which u is a tangent. We get that result, if in u we replace t by its value, in terms of x and y, derived from the equation $\frac{du}{dt}=0$. Now, if we differentiate, we have $\frac{du}{dx}=\left(\frac{du}{dx}\right)+\frac{du}{dt}\frac{dt}{dx}$ and $\frac{du}{dy}=\left(\frac{du}{dy}\right)+\frac{du}{dt}\frac{dt}{dy}$, where $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$ are the differentials of u on the supposition that t is constant. And since $\frac{du}{dt}=0$ it is evident that $\frac{du}{dx}$, $\frac{du}{dy}$ are the same as on the supposition that t is constant. It follows that the eliminant in question denotes a curve touched by u.

If it be required to draw a tangent to this curve through any point, we have only to substitute the coordinates of that point in the equation $u=0$, and determine t so as to satisfy that equation. This problem will have a definite number of solutions, and the number will plainly be the number of tangents which can be drawn to the curve from an arbitrary point; that is to say, the class of the curve. For example, the envelope of the line

$$at^3+3bt^2+3ct+d=0,$$

where a, b, c, d, are linear functions of the coordinates, is plainly a curve of the third class.

321. Now let us proceed in like manner with a surface. The equation of the tangent plane to a surface is a function of the three coordinates, which being connected by only one relation (viz. the equation of the surface), the equation of the tangent plane, when most simplified, contains two variable parameters. The converse problem is to find the envelope of a plane whose equation $u=0$ contains two variable parameters s, t. The equation of any other plane answering to the values $s+h$, $t+k$ will be

$$u+\left(h\frac{du}{ds}+k\frac{du}{dt}\right)+\frac{1}{1.2}\left(h^2\frac{d^2u}{ds^2}+\&c.\right)+\&c.=0.$$

Now, in the limit, when h and k are taken indefinitely small, they may preserve any finite ratio to each other, $k=\lambda h$. We see thus that the intersection of any plane by a consecutive one is not a definite line, but may be any line represented by the equations $u=0$, $\dfrac{du}{ds}+\lambda\dfrac{du}{dt}=0$, where λ is indeterminate. But we see also that all planes consecutive to u pass through the *point* given by the equations $u=0$, $\dfrac{du}{ds}=0$, $\dfrac{du}{dt}=0$.

From these three equations we can eliminate the parameters s, t, and so find the locus of all those points where a plane of the system is met by the series of consecutive planes. It is proved, as in the last article, that the surface represented by this eliminant is touched by u. If it be required to draw a tangent plane to this surface through any point, we have only to substitute the coordinates of that point in the equation $u=0$. The equation then containing two indeterminates s and t can be satisfied in an infinity of ways; or, as we know, through a given point an infinity of tangent planes can be drawn to the surface, these planes enveloping a cone.

Suppose, however, that we either consider t as constant, or as any definite function of s, the equation of the tangent plane is reduced to contain a single parameter, and the envelope of those particular tangent planes which satisfy the assumed condition is a developable. Thus, again, we may see

the *reciprocity between a developable and a curve.* When a surface is considered as the locus of a number of points connected by a given relation, if we add another relation connecting the points we obtain a curve traced on the given surface. So when we consider a surface as the envelope of a series of planes connected by a single relation, if we add another relation connecting the planes we obtain a developable enveloping the given surface.

[Ex. In Art. 316 four methods A, B, C, D were described of representing a curve in space by means of equations. What are the reciprocal methods of representing developables?]

322. Let us now see *what properties of developables are to be deduced from considering the developable as the envelope of a plane whose equation contains a single variable parameter.* In the first place it appears that through any assumed point can be drawn, not, as before, an infinity of planes of the system forming a cone, but a definite number of planes. Thus, if it be required to find the envelope of $at^3 + 3bt^2 + 3ct + d$, where a, b, c, d represent planes, it is obvious that only three planes of the system can be drawn through a given point, since on substituting the coordinates of any point we get a cubic for t. Again, any plane of the system is cut by a consecutive plane in a definite line; namely, the line $u = 0$, $\frac{du}{dt} = 0$; and if we *eliminate t between these two equations, we obtain the surface generated by all those lines, which is the required developable.*

It is proved, as at Art. 320, that the plane u touches the developable at every point which satisfies the equations $u = 0$, $\frac{du}{dt} = 0$; or, in other words, touches along the whole of the line of the system corresponding to u. It was proved (Art. 110) that in general when a surface contains a right line the tangent plane at each point of the right line is different. But in the case of the developable the tangent plane at every point is the same. If x be the plane which touches all along the

line xy, the equation of the surface can be thrown into the form $x\phi + y^2\psi = 0$ (see Art. 110).*

323. Let us now consider *three consecutive planes of the system*, and it is evident, as before, that their intersection satisfies the equations $u=0$, $\frac{du}{dt}=0$, $\frac{d^2u}{dt^2}=0$. For any value of t, the point is thus determined where any line of the system is met by the next consecutive. The locus of these points is got by eliminating t between these equations. We thus obtain two equations in x, y, z, one of them being the equation of the developable. These two equations represent a curve traced on the developable. Thus it is evident that, starting with the definition of a developable as the envelope of a moveable plane, we are led back to its generation as the locus of tangents to a curve. For the consecutive intersections of the planes form a series of lines, and the consecutive intersections of the lines are a series of points forming a curve to which the lines are tangents. We shall presently show that the curve is a *cuspidal edge* † *on the developable.*

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* It seems unnecessary to enter more fully into the subject of envelopes in general, since what is said in the text applies equally if u, instead of representing a plane, denote any surface whose equation includes a variable parameter. Monge calls the curve $u=0$, $\frac{du}{dt}=0$, in which any surface of the system is intersected by the consecutive, the *characteristic* of the envelope. For the nature of this curve depends only on the manner in which the variables x, y, z enter into the function u, and not on the manner in which the constants depend on the parameter. Thus, when u represents a plane, the characteristic is always a right line, and the envelope is the locus of a system of right lines. When u represents a sphere, the characteristic being the intersection of two consecutive spheres is a circle, and the envelope is the locus of a system of circles. And so envelopes in general may be divided into families according to the nature of the characteristic.

† Monge has called this the "arête de rebroussement," or "edge of regression" of the developable. There is a similar curve on every envelope, namely, the locus of points in which each "characteristic" is met by the next consecutive. The part of the characteristic on one side of this curve generates one sheet of the envelope, and that on the other side generates another sheet. The two sheets touch along this curve which is their common limit, and is a

324. Four consecutive planes of the system will not meet in a point unless the four conditions be fulfilled $u=0$, $\frac{du}{dt}=0$, $\frac{d^2u}{dt^2}=0$, $\frac{d^3u}{dt^3}=0$. It is in general possible to find certain values of t, for which these equations will be satisfied. For if we eliminate x, y, z, we get the condition that the four planes, whose equations have been just written, shall meet in a point. Since this condition expresses that a function of t is equal to nothing, we shall in general get a determinate number of values of t for which it is satisfied. There are therefore in general a certain number of points of the system through which four planes of the system pass; or, in other words, a certain number of points in which three consecutive lines of the system intersect. We shall call these, as at *Higher Plane Curves*, p. 25, the *stationary points* of the system; since in this case the point determined as the intersection of two consecutive lines coincides with that determined as the intersection of the next consecutive pair. A stationary point is a cusp on the cuspidal edge.

Reciprocally, there will be in general a certain number of planes of the system which may be called *stationary planes*. These are the planes which contain four consecutive points of the system; for, in such a case, the planes 123, 234 evidently coincide.

[A stationary plane meets *any* plane in an inflexional tangent of the section of the developable by the latter plane. At an ordinary point of the system the cuspidal curve crosses the "plane" of the system (the osculating plane) at the point, but a stationary plane does not in general cross it. This will subsequently be proved.

If for some value of t, the planes $u=0$, $\frac{du}{dt}=0$, $\frac{d^2u}{dt^2}=0$, have a common line, this will be a *stationary generator* on the developable, and the point where it meets the cuspidal edge will be an inflexion thereon. A stationary generator meets any plane section in a cusp. It reciprocates into a stationary generator.

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cuspidal edge of the envelope. Thus, in the case of a cone, the parts of the generating lines on opposite sides of the vertex generate opposite sheets of the cone, and the cuspidal edge in this case reduces itself to a single point, namely, the vertex.

If a system be defined tangentially by expressing the "planes" as general functions of a parameter, stationary points are, as we have seen, ordinary singularities. But stationary planes are exceptional, since two conditions must be satisfied by the constants if $u = 0$, and $u + \frac{du}{dt} = 0$ represent the same plane. On the other hand, if the system be originally defined by expressing the "points" as general functions of a parameter, stationary planes are ordinary singularities and stationary points are exceptional. In both cases a stationary generator is exceptional.

Ex. 1. Find the equation of the developable whose planes are of the form

$$xt^3 + 3yt^2 + 3zt + 1 = 0.$$

Prove that the cuspidal edge is the partial intersection of $xz = y^2$, $y = z^2$, and show that its coordinates may be expressed in the form

$$x = \theta^3,\ y = \theta^2,\ z = \theta.$$

Prove that it has no stationary point.

Ex. 2. Prove that the point $x = y = z = w$ is a stationary point on the system

$$xt^4 + 4yt^3 + 6zt^2 + (4t + 1)\,w = 0,$$

x, y, z, w being planes, and that the tangent plane at the point is

$$x - 4y + 6z - 3w = 0.$$

Ex. 3. For the system

$$xt^4 + 6yt^2 + 4zt + w = 0$$

the point $y = z = w = 0$ is a stationary point, and $w = 0$ is the tangent plane thereat ($t = 0$).

For the same system the plane $x = 0$ is a stationary plane ($t = \infty$) and $x = y = z = 0$ is the point where it meets the cuspidal edge.

Ex. 4. For the system

$$xt^4 + 4yt^3 + 6\lambda wt^2 + 4zt + w = 0$$

the line $z = 0$, $w = 0$ is a stationary generator ($t = 0$), meeting the cuspidal edge in a cusp where $y = 0$.

If $\lambda = 0$, the line $x = 0$, $y = 0$ is also a stationary generator ($t = \infty$) and the point $x = y = z = 0$ is the corresponding cusp.

Ex. 5. Prove that the reciprocal of the system in Ex. 3 is of the same form. This is proved by expressing the coordinates of the cuspidal edge in the form

$$x = a\theta^4,\ y = b\theta^2,\ z = c\theta,\ w = 1.]$$

325. We proceed to show how, from Plücker's equations connecting the ordinary singularities of plane curves,*

* These equations are as follow: see *Higher Plane Curves*, p. 65. Let μ be the degree of a curve, ν its class, δ the number of its double points, τ that of its double tangents, κ the number of its cusps, ι that of its points of inflexion; then

$$\nu = \mu(\mu - 1) - 2\delta - 3\kappa;\quad \mu = \nu(\nu - 1) - 2\tau - 3\iota;$$
$$\iota = 3\mu(\mu - 2) - 6\delta - 8\kappa;\quad \kappa = 3\nu(\nu - 2) - 6\tau - 8\iota.$$

Whence also $\iota - \kappa = 3(\nu - \mu)$; $2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9)$.

Cayley* has deduced equations connecting the ordinary singularities of developables. We shall first make an enumeration of these singularities. We speak of the "points of the system," the "lines of the system," and the "planes of the system" as explained (Art. 123).

Let m be the number of points of the system which lie in any plane; or, in other words, the *degree* of the curve which generates the developable.

Let n be the number of planes of the system which can be drawn through an arbitrary point. We have proved (Art. 322) that the number of such planes is definite. We shall call this number the *class* of the system.

Let r be the number of lines of the system which intersect an arbitrary right line. It is plain that if we form the condition that u, $\frac{du}{dt}$, and any assumed right line may intersect, the result will be an equation in t, which gives a definite number of values of t. Let r be the number of solutions of this equation. We shall call this number the *rank* of the system (or the *order* of the developable), and we shall show that all other singularities of the system can be expressed in terms of the three just enumerated.

Let α be the number of stationary planes, and β the number of stationary points (Art. 324).

Two non-consecutive lines of the system may intersect. When this happens we call the point of meeting a "point on two lines," and their plane a "plane through two lines". Let x be the number of "points on two lines" which lie in a given plane, and y the number of "planes through two lines" which pass through a given point.

In like manner we shall call the line joining any two points of the system a "line through two points," and the intersection of any two planes a "line in two planes". Let g be the number of "lines in two planes" which lie in a given

* See Liouville's *Journal*, Vol. X. p. 245; *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 18.

plane, and h the number of "lines through two points" which pass through a given point. The number h may also be called the number of *apparent* double points of the curve; for to an eye placed at any point, two branches of the curve appear to intersect if any line drawn through the eye meet both branches.

The developable has other singularities which will be considered in a subsequent chapter, but these are the singularities which Plücker's equations (note, Art. 325) enable us to determine.

326. Consider now *the section of the developable by any plane.* It is obvious that the points of this curve are the traces on its plane of the "lines of the system," while the tangent lines of the section are the traces on its plane of the "planes of the system". The degree of the section is therefore r, since it is equal to the number of points in which an arbitrary line drawn in its plane meets the section, and we have such a point whenever the line meets a "line of the system".

The class of the section is plainly n. For the number of tangent lines to the section drawn through an arbitrary point is evidently the same as the number of "planes of the system" drawn through the same point.

A double point on the section will arise whenever two "lines of the system" meet the plane of section in the same point. The number of such points by definition is x. The tangent lines at such a double point are usually distinct, because the two planes of the system corresponding to the lines of the system intersecting in any of the points x are commonly different.

The number of double tangents to the section is in like manner g; since a double tangent arises whenever two planes of the system meet the plane of section in the same line.

The m points of the system which lie in the plane of section are cusps of the section. For each is a double point as

being the intersection of two lines of the system; and the tangent planes at these points coincide, since the two consecutive lines, intersecting in one of the points m, lie in the same plane of the system. This proves, what we have already stated, that *the curve whose tangents generate the developable is a cuspidal edge on the developable;* for it is such that every plane meets that surface in a section which has as cusps the points where the same plane meets the curve.

Lastly, we get a point of inflexion (or a stationary tangent) wherever two consecutive planes of the system coincide. The number of the points of inflexion is therefore a.

We are to substitute, then, in Plücker's formulæ,

$$\mu = r,\ \nu = n,\ \delta = x,\ \tau = g,\ \kappa = m,\ \iota = a.$$

And we have

$$n = r(r-1) - 2x - 3m;\quad r = n(n-1) - 2g - 3a,$$
$$a = 3r(r-2) - 6x - 8m;\quad m = 3n(n-2) - 6g - 8a,$$

whence also

$$m - a = 3(r-n);\quad 2(x-g) = (r-n)(r+n-9).$$

327. The reciprocal system of equations is found by considering *the cone whose vertex is any point and which stands on the given curve.* It appears at once by considering the section of a cone by any plane that the same equations connect the double edges, double tangent planes, &c., of cones, which connect the double points, double tangents, &c., of plane curves.

The edges of the cone which we are now considering are the lines joining the vertex to all the points of the system; and the tangent planes to the cone are the planes connecting the vertex with the lines of the system, for evidently the plane containing two consecutive edges of the cone must contain the line joining two consecutive points of the system.

The degree of the cone is plainly the same as the degree of the curve, and is therefore m.

The class of the cone is the same as the number of tangent planes to the cone which pass through an arbitrary line drawn through the vertex. Now since each tangent plane contains

a line of the system, it follows that we have as many tangent planes passing through the arbitrary line as there are lines of the system which meet that line. The number sought is therefore r.

It is easy to see that the class of this cone is the same as the degree of the developable which is the reciprocal of the points of the given system. Hence, *the degree of the developable generated by the tangents to any curve is the same as the degree of the developable which is the reciprocal of the points of that curve*, see note, Art. 124.

A double edge of the cone arises when the same edge of the cone passes through two points of the system, or $\delta = h$. The tangent planes along that edge are the planes joining the vertex to the lines of the system which correspond to each of these points.

A double tangent plane will arise when the same plane through the vertex contains two lines of the system, or $\tau = y$.

A stationary or cuspidal edge of the cone will only exist when there is a stationary point in the system, or $\kappa = \beta$.

Lastly, a stationary tangent plane will exist when a plane containing two consecutive lines of the system passes through the vertex, or $\iota = n$.

Thus we have $\mu = m$, $\nu = r$, $\delta = h$, $\tau = y$, $\kappa = \beta$, $\iota = n$. Hence, by the formulæ (note, Art. 325),

$$r = m(m-1) - 2h - 3\beta\,;\quad m = r(r-1) - 2y - 3n\,;$$
$$n = 3m(m-2) - 6h - 8\beta\,;\quad \beta = 3r(r-2) - 6y - 8n.$$

Whence also

$$n - \beta = 3(r-m)\,;\quad 2(y-h) = (r-m)(r+m-9).$$

And combining these equations with those found in the last article, we have also

$$a - \beta = 2(n-m)\,;\quad x - y = n - m\,;\quad 2(g-h) = (n-m)(n+m-7).$$

Plücker's equations enable us, when three of the singularities of a plane curve are given, to determine all the rest. Now three quantities r, m, n are common to the equations of this and of the last article. Hence, *when any three of the singularities which we have enumerated, of a curve in space, are given, all the rest can be found.*

22*

328. It is to be observed that, besides the singularities which we have enumerated, a curve may have others which may be treated by means of Plücker's equations. It may, for example, besides its apparent double points, have H actual double points or nodes; viz., considering the curve as generated by the motion of a variable point, we have a node if ever the point comes twice into the same position. Reciprocally, the system may have G double planes; viz., considering the developable as the envelope of a plane, if in the course of its motion the plane comes twice into the same position, we have a double plane. These singularities will be taken into account if, in the formulæ of Art. 326, we write $\tau = g + G$ instead of $\tau = g$, and in the formulæ of Art. 327, write $\delta = h + H$. In like manner, the system may have v stationary lines, or lines containing three consecutive points of the system. Such a line meets in a cusp the section of the developable by any plane, and accordingly, in Art. 326, instead of having $\kappa = m$, we have $\kappa = m + v$; and, in like manner, in Art. 327, instead of $\iota = n$, we have $\iota = n + v$. Once more the system may have ω double lines, or lines containing each two pairs of consecutive points of the system. Taking these into account we have, in Art. 326, $\delta = x + \omega$, and in Art. 327, $\tau = y + \omega$.

329. To illustrate this theory, let us take the *developable which is the envelope of the plane*

$$at^k + kbt^{k-1} + \tfrac{1}{2}k\,(k-1)\,ct^{k-2} + \&c. = 0,$$

where t is a variable parameter, a, b, c, &c., represent planes, and k is any integer.

The class of this system is obviously k, and the equation of the developable being the discriminant of the preceding equation, its degree is $2\,(k-1)$; hence $r = 2\,(k-1)$.

Also it is easy to see that this developable can have no stationary planes. For, in general, if we compare coefficients in the equations of two planes, three conditions must be satisfied in order that the two planes may be identical. If then we attempt to determine t so that any plane may be identical with the consecutive one, we find that we have

three conditions to satisfy, and only one constant t at our disposal.

Having then $n=k$, $r=2(k-1)$, $\alpha=0$, the equations of the last two articles enable us to determine the remaining singularities. The result is

$$m=3\,(k-2)\,,\ \beta=4\,(k-3)\,,\ x=2\,(k-2)\,(k-3)\,,$$
$$y=2\,(k-1)\,(k-3)\,,\ g=\tfrac{1}{2}\,(k-1)\,(k-2)\,,\ h=\tfrac{1}{2}\,(9k^2-53k+80).$$

The greater part of these values can be obtained independently, see *Higher Plane Curves*, p. 71. But in order to economize space we do not enter into details.

[In like manner we may consider the reciprocal problem, viz., to determine the singularities of a developable when the coordinates of its cuspidal edge are integral rational functions of degree k in a variable parameter.]

330. The case considered in the last article, which is that when the variable parameter enters only rationally into the equation, enables us to verify easily many properties of developables. Since the system $u=0$, $\dfrac{du}{dt}=0$ is obviously reducible to

$$at^{k-1}+(k-1)\,bt^{k-2}+\&c.=0,\quad bt^{k-1}+(k-1)\,ct^{k-2}+\&c.=0,$$

and the system $u=0$, $\dfrac{du}{dt}=0$, $\dfrac{d^2u}{dt^2}=0$ is reducible to

$$at^{k-2}+(k-2)\,bt^{k-3}+\&c.=0,\ bt^{k-2}+(k-2)\,ct^{k-3}+\&c.=0,$$
$$ct^{k-2}+(k-2)\,dt^{k-3}+\&c.=0,$$

it follows that a is itself a plane of the system (namely that corresponding to the value $t=\infty$), ab is the corresponding line, and abc the corresponding point. Now we know from the theory of discriminants (see *Higher Algebra*, Art. 111) that the equation of the developable is of the form $a\phi+b^2\psi=0$, where ψ is the discriminant of u when in it a is made $=0$. Thus we verify what was stated (Art. 322) that a touches the developable along the whole length of the line ab. Further, ψ is itself of the form $b\phi+c^2\psi$. If now we consider the section of the developable by one of the planes of the system (or, in other words, if we make $a=0$ in the equation of the developable), the section consists of the line ab

twice and of a curve of the degree $r-2$; and this curve (as the form of the equation shows) touches the line ab at the point abc, and consequently meets it in $r-4$ other points. These are all "points on two lines," being the points where the line ab meets other lines of the system. And it is generally true that if r be the rank of a developable *each line of the system meets $r-4$ other lines of the system.* The locus of these points forms a double curve on the developable, the degree of this curve is x, and other properties of it will be given in a subsequent chapter, where we shall also determine certain other singularities of the developable.

We add here a table of the singularities of some special sections of the developable. The reader, who may care to examine the subject, will find no great difficulty in establishing them. I have given the proof of the greater part of them, *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 24. See also Cayley's Paper, *Quarterly Journal*, Vol. XI. p. 295.

Section by a plane of the system

$$\mu=r-2,\ \nu=n-1,\ \iota=a,\ \kappa=m-3,\ \tau=g-n+2,\ \delta=x-2r+8.$$

Cone whose vertex is a point of the system

$$\mu=m-1,\ \nu=r-2,\ \iota=n-3,\ \kappa=\beta,\ \tau=y-2r+8,\ \delta=h-m+2.$$

Section by plane passing through a line of the system

$$\mu=r-1,\ \nu=n,\ \iota=a+1,\ \kappa=m-2,\ \tau=g-1,\ \delta=x-r+4.$$

Cone whose vertex is on a line of the system

$$\mu=m,\ \nu=r-1,\ \iota=n-2,\ \kappa=\beta+1,\ \tau=y-r+4,\ \delta=h-1.$$

Section by plane through two lines

$$\mu=r-2,\ \nu=n,\ \iota=a+2,\ \kappa=m-4,\ \tau=g-2,\ \delta=x-2r+9.$$

Cone whose vertex is a point on two lines

$$\mu=m,\ \nu=r-2,\ \iota=n-4,\ \kappa=\beta+2,\ \tau=y-2r+9,\ \delta=h-2.$$

Section by a stationary plane

$$\mu=r-3,\ \nu=n-2,\ \iota=a-1,\ \kappa=m-4,\ \tau=g-2n+6,\ \delta=x-3r+13.$$

Cone whose vertex is a stationary point

$$\mu=m-2,\ \nu=r-3,\ \iota=n-4,\ \kappa=\beta-1,\ \tau=y-3r+13,\ \delta=h-2m+6.$$

In the preceding we have not taken account of the singularities G, H, v, ω, having shown in Art. 328 how to modify the formulæ so as to include them. The following formulæ of Cayley's relate to these singularities:—

Section by a plane G

$$\mu=r-4,\ \nu=n-2,\ \iota=a,\ \kappa=m-6+v,\ \tau=g-2n+6+G-1,\ \delta=x-4r+20+\omega.$$

Cone whose vertex is a point H

$\mu=m-2,\ \nu=r-4,\ \iota=n-6+v,\ \kappa=\beta,\ \tau=y-4r+20+\omega,\ \delta=h-2m+6+H-1.$

Section by plane through stationary line v

$\mu=r-2,\ \nu=n,\ \iota=a+2,\ \kappa=w-3+v-1,\ \tau=g-2+G,\ \delta=x-2r+9+\omega.$

Cone whose vertex is on the stationary line v

$\mu=m,\ \nu=r-2,\ \iota=n-3+v-1,\ \kappa=\beta+2,\ \tau=y-2r+9+\omega,\ \delta=h-2+H.$

Section by tangent plane at contact of line v

$\mu=r-3,\ \nu=n-1,\ \iota=a+1,\ \kappa=m-4+v-1,\ \tau=g-n+1+G,\ \delta=x-3r+14+\omega.$

Cone whose vertex is contact of line v

$\mu=m-1,\ \nu=r-3,\ \iota=n-4+v-1,\ \kappa=\beta+1,\ \tau=y-3r+14+\omega,\ \delta=h-m+1+H.$

Section by plane through double tangent ω

$\mu=r-2,\ \nu=n,\ \iota=a+2,\ \kappa=m-4+v,\ \tau=g-2+G,\ \delta=x-2r-10+\omega-1.$

Cone whose vertex is on double tangent ω

$\mu=m,\ \nu=r-2,\ \iota=n-4+v,\ \kappa=\beta+2,\ \tau=y-2r+10+\omega-1,\ \delta=h-2+H.$

Section by tangent plane at one of the contacts of line ω

$\mu=r-3,\ \nu=n-1,\ \iota=a+1,\ \kappa=m-5+v,\ \tau=g-n+1+G,\ \delta=x-3r+15+\omega-1.$

Cone whose vertex is a contact of line ω

$\mu=m-1,\ \nu=r-3,\ \iota=n-5+v,\ \kappa=\beta+1,\ \tau=y-3r+15+\omega-1,\ \delta=h-m+1+H.$

Section II. Classification of Curves.

331. The following enumeration rests on the principle that a curve of the degree r meets a surface of the degree p in pr points. This is evident when the curve is the complete intersection of two surfaces whose degrees are m and n. For then we have $r=mn$ and the three surfaces intersect in mnp points. It is true also by definition when the surface breaks up into p planes. We shall assume, for the moment, that the principle is generally true.

The use we make of the principle is this. Suppose that we *take on a curve of the degree r as many points as are sufficient to determine a surface of the degree p; then if the number of points so assumed be greater than pr, the surface described through the points must altogether contain the curve;* for otherwise the principle would be violated.

We assume in this that the curve is a *proper* curve of the degree r, for if we took two curves of the degree m and n (where $m+n=r$), the two together might be regarded as a complex curve of the degree r, and if *either* lay altogether on any surface of the degree p, of course we could take on that curve any number of points common to the curve and surface.

All this will be sufficiently illustrated by the examples which follow.

[The curves with which Salmon deals in this section are *defined* as the partial or complete intersections of two surfaces of finite degree. Such curves and surfaces may be called *algebraic*, since they meet planes and straight lines, respectively, in points defined by algebraic numbers, i.e., solutions, real or imaginary, of equations of finite degree.* A complete intersection of two surfaces is defined algebraically but a partial intersection (B) is only known if we know the residual or remaining intersection; e.g., a cubic curve is known if it is defined as the partial intersection of two quadrics with a common generator. The residual must again be defined either as the complete intersection of two surfaces or as the residual of a known curve. We proceed in this way until B is ultimately defined in terms of complete intersections, which may of course consist partly of plane algebraic curves.

Using these modes of defining a curve we can prove the principle stated above. The degree of a curve is *defined* as the number of points in which it meets a plane; we have to prove that it meets a surface of degree r in r times this number of points. It is evidently true if the curve is a complete intersection. Now let A and B together be the complete intersection (C) of two surfaces of degree m and n. We can show that if the principle holds for B it also holds for A. Let a, b, c be the degrees of the curves A, B, C. Then $c = a + b$, and a surface U of degree r meets C in $(a + b)r$ points. But by hypothesis it meets B in br points, therefore it meets A in ar points. Assuming that B is ultimately definable, in the manner just indicated, by means of complete intersections, we see, by carrying on this argument, that the principle is proved by mathematical induction.]

332. *There is no line of the first degree but the right line.* For through any two points of a line of the first degree and any assumed point we can describe a plane which must altogether contain the line, since otherwise we should have a line of the first degree meeting the plane in more points than one. In like manner we can draw a second plane containing the line, which must therefore be the intersection of two planes; that is to say, a right line.

There is no proper line of the second degree but a conic. Through any three points of the line we can draw a plane, which the preceding reasoning shows must altogether contain the line. The line must therefore be a plane curve of the second degree.

* We assume in this discussion that our surfaces and curves are algebraic.

The exception noted in the last article would occur if the line of the second degree consisted of two right lines not in the same plane; for then the plane through three points of the system would only contain *one* of the right lines. In what follows we shall not think it necessary to notice this again, but shall speak only of proper curves of their respective orders.

333. *A curve of the third degree must either be a plane cubic or the partial intersection of two quadrics*, as explained, Art. 315.*

For through seven points of the curve and any two other points describe a quadric; and, as before, it must altogether contain the curve. If the quadric break up into two planes, the curve may be a plane curve lying in one of the planes. As we may evidently have plane curves of any degree we shall not think it necessary to notice these in subsequent cases. If then the quadric do not break up into planes, we can draw a second quadric through the seven points, and the intersection of the two quadrics includes the given cubic. The complete intersection being of the fourth degree, it must be the cubic together with a right line; it is proved therefore that the only non-plane cubic is that explained, Art. 315.

[333*a*. Since two quadric cones can be drawn through the intersection of two quadrics with a common generator (Art. 202), we may always replace the quadrics in the above proposition by cones, which evidently have a common generator.

The coordinates of any point on a twisted cubic can be expressed as cubic functions of a parameter t. Let $ABCD$ be the tetrahedron of reference, U and

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* Non-plane curves of the third degree appear to have been first noticed by Möbius in his *Barycentric Calculus*, 1827. Some of their most important properties are given by Chasles in Note XXXIII. to his *Aperçu Historique*, 1837, and in a paper in Liouville's *Journal* for 1857, p. 397. More recently the properties of these curves have been treated by Schröter, *Crelle*, Vol. LVI., and by Cremona, of Milan, *Crelle*, Vol. LVIII., p. 138. Considerable use has been made of the latter paper in the articles which immediately follow. [A list of the principal papers on cubic curves published before 1907 will be found in Loria, *Il Passato ed il Presente delle principali Teorie Geometriche*, third edition, p. 183 ff., p. 379 ff.]

V the two cones, A the vertex of U and D that of V; then AD is the common generator $y = 0$, $z = 0$. Let ACD ($y = 0$) and ABD ($z = 0$) be the tangent planes to U and V, respectively, along AD, and let AB be a generator of U and CD of V. Let ABC ($w = 0$) be the tangent plane to U along AB, and BCD the tangent plane to V along CD. Then the cones $U = 0$, $V = 0$ may be written

$$z^2 - yw = 0,\ y^2 - zx = 0$$

and the coordinates of any point on their curve of intersection (the points on the common generator excepted) may be written

$$x = t^3,\ y = t^2,\ z = t,\ w = 1.$$

By linear transformation it follows that the coordinates referred to any planes are cubic functions of t.

The cubic is evidently the intersection of corresponding planes of the three homographic pencils of planes

$$x = yt,\ y = zt,\ z = wt.$$

Since a system of quadrics having a common generator reciprocates into a like system (generators being self-reciprocal), we see that *the coordinates of the tangent planes of the circum developable circumscribing two quadrics with a common generator are cubic functions of a parameter.* The two cones mentioned above reciprocate into two conics, both touching the line of intersection of their planes (a limiting case of two quadrics with a common generator), and the developable is simply the common tangent developable of these two conics.

Hence by means of two conics touching the line of intersection of their planes we can construct a developable of this type. Let the tangent line at any point P on the first conic meet the line of intersection in O, and let OQ be the tangent line to the second conic at Q; then PQ is a generator of the developable, the plane POQ being the tangent plane.

Ex. If two quadrics have a common generator, it meets their remaining curve of intersection in two points at each of which the quadrics have the same tangent plane. These tangent planes are common to all quadrics having the common generator and the same curve of intersection. The system is represented by $\lambda\,(y^2 - zx) + \mu(z^2 - yw) = 0$ and the two points are given by $t = 0$, $t = \infty$.]

334. The cone containing a curve of the m^{th} degree and whose vertex is a point on the curve, is of the degree $m - 1$ since it cuts a plane through the vertex in $m - 1$ generators; hence the cone containing a cubic, and whose vertex is on the curve, is of the second degree. *We can thus describe a twisted cubic through six given points.* For we can describe a cone of the second degree of which the vertex and five edges are given, since evidently we are thus given five points in the section of the cone by any plane, and can thus determine that section. If then we are given six points, a, b, c, d, e, f, we can describe a cone having the point a for vertex, and

the lines ab, ac, ad, ae, af for edges; and in like manner a cone having b for vertex and the lines ba, bc, bd, be, bf for edges. The intersection of these cones consists of the common edge ab and of a cubic which is the required curve passing through the six points.

The theorem that the lines joining six points of a cubic to any seventh are edges of a quadric cone, leads at once to the following by Pascal's theorem: "The lines of intersection of the planes 712, 745; 723, 756; 734, 761 lie in one plane". Or, in other words, "the points where the planes of three consecutive angles 567, 671, 712 meet the opposite sides lie in one plane passing through the vertex 7".* Conversely if this be true for two vertices of a heptagon it is true for all the rest; for then these two vertices are vertices of cones of the second degree containing the other points, which must therefore lie on the cubic which is the intersection of the cones.

335. *A cubic traced on a hyperboloid of one sheet meets all its generators of one system once, and those of the other system twice.*

Any generator of a quadric meets in two points its curve of intersection with any other quadric, namely, in the two points where the generator meets the other quadric. Now when the intersection consists of a right line and a cubic, it is evident that the generators of the same system as the line, since they do not meet the line, must meet the cubic in the two points; while the generators of the opposite system, since they meet the line in one point, only meet the cubic in one other point.

Conversely we can describe a system of hyperboloids through a cubic and any chord which meets it twice. For, take seven points on the curve, and an eighth on the chord joining any two of them; then through these eight points an infinity of quadrics can be described. But since three of

* Cremona adds, that when the six points are fixed and the seventh variable, this plane passes through a fixed chord of the cubic.

these points are on a right line, that line must be common to all the quadrics, as must also the cubic on which the seven points lie.

[Ex. State the reciprocal of the theorem of this article.]

336. The question to find the envelope of $at^3 - 3bt^2 + 3ct - d$ (where a, b, c, d represent planes and t is a variable parameter) is a particular case of that discussed, Art. 329. We have

$$r = 4,\ m = n = 3,\ a = \beta = 0,\ x = y = 0,\ g = h = 1.$$

Thus the *system is of the same nature as the reciprocal system*, and all theorems respecting it are consequently two-fold. The system being of the third degree must be of the kind we are considering; and this also appears from the equation of the envelope

$$(ad - bc)^2 = 4\ (b^2 - ac)\ (c^2 - bd),$$

for it is easy to see that any pair of the surfaces $ad - bc$, $b^2 - ac$, $c^2 - bd$, have a right line common, while there is a cubic common to all three, which is a double line on the envelope.

[Conversely (Art. 333a) all systems of the third degree (m) or class (n) are of this kind.]

It appears from the table just given that every plane contains one "line in two planes," or that *the section of the developable by any plane has one double tangent;* while reciprocally through any point can be drawn one line to meet the cubic twice; the cone therefore, whose vertex is that point, and which stands on the curve, has one double edge; for, in other words, *the cubic is projected on any plane into a cubic having a double point.* [This also follows from the fact that the coordinates of the cubic are rational functions of a parameter.]

The three points of inflexion of a plane cubic are in one right line. Now it was proved (Art. 327) that the points of inflexion correspond to the three planes of the system which can be drawn through the vertex of the cone. Hence *the three points of the system, which correspond to the three planes*

*which can be drawn through any point O, lie in one plane passing through that point.**

Further, it is known that when a plane cubic has a conjugate point, its three points of inflexion are real; but that when the cubic has a double point, the tangents at which are real, then two of the points of inflexion are imaginary. Hence, if the chord which can be drawn through any point O meet the cubic in two real points, then two of the planes of the system which can be drawn through O are imaginary. Reciprocally, if through any line two real planes of the system can be drawn, then any plane through that line meets the curve in two imaginary points, and only one real one.†

[Ex. The system being written $x\theta^3 + 3y\theta^2 + 3z\theta + w = 0$ prove that the ray coordinates of a "line of the system" through θ are $\theta^2 : 2\theta : 1 : 3\theta^2 : -2\theta^3 : \theta^4$.]

337. These theorems can also be easily established algebraically; for the point of contact of the plane $at^3 - 3bt^2 + 3ct - d$, being given by the equations $at = b$, $bt = c$, $ct = d$, may be denoted by the coordinates $a = 1$, $b = t$, $c = t^2$, $d = t^3$. Now the three values of t answering to planes passing through any point are given by the cubic $a't^3 - 3b't^2 + 3c't - d' = 0$, whence it is evident, from the values just found, that the points of contact lie in the plane $a'd - 3b'c + 3c'b - d'a = 0$. But this plane passes through the given point. Hence *the intersection of three planes of the system lies in the plane of the corresponding points.* The equation just written is unaltered if we interchange accented and unaccented letters. Hence, *if a point A be in the plane of points of contact, corresponding to any point B, B will be in the plane in like manner corresponding to A.* And again, the planes which thus correspond to all the points of a line AB pass through a fixed right line, namely, the intersection of the planes corresponding to A and B. The relation between the lines is evidently reciprocal. To any plane of the system will correspond in this sense the corresponding point of the system;

* Chasles, *Liouville*, 1857. Schröter, *Crelle*, Vol. LVI.

† Joachimsthal, *Crelle*, Vol. LVI. p. 45. Cremona, *Crelle*, Vol. LVIII. p. 146.

and to a line in two planes corresponds a chord joining two points.

The three points where any plane $Aa + Bb + Cc + Dd$ meets the curve have their t's given by the equation $Dt^3 + Ct^2 + Bt + A = 0$, and when this is a perfect cube, the plane is a plane of the system. From this it follows at once, as Joachimsthal has remarked, that any plane drawn through the intersection of two real planes of the system meets the curve in but one real point. For, in such a case, the cubic just written is the sum of two cubes and has but one real factor.

338. We have seen (Art. 134) that the locus of the poles of a fixed plane with regard to a system of quadrics having a common curve is a twisted cubic. More generally, such a curve is expressed by the result of the elimination of λ between the system of equations $\lambda a = a'$, $\lambda b = b'$, $\lambda c = c'$. Now since the anharmonic ratio of four planes, whose equations are of the form $\lambda a = a'$, $\lambda' a = a'$, &c., depends only on the coefficients λ, λ', &c. (see *Conics*, Art. 59), this mode of obtaining the equation of the cubic may be interpreted as follows: *Let there be a pencil of planes through any line aa', a homographic pencil through any other line bb', and a third through cc', then the locus of the intersection of three corresponding planes of the pencils is a twisted cubic.* The lines aa', bb', cc' are evidently lines through two points, or chords of the cubic. Reciprocally, *if three right lines be homographically divided, the plane of three corresponding points envelopes the developable generated by a twisted cubic*, and the three right lines are "lines in two planes" of the system.

The line joining two corresponding points of two homographically divided lines touches a conic when the lines are in one plane, and generates a hyperboloid when they are not. Hence, given a series of points on a right line and a homographic series either of tangents to a conic or of generators of a hyperboloid, the planes joining each point to the corresponding line envelope a developable, as above stated.

Ex. If the four faces of a tetrahedron pass through four fixed lines and the three edges of a face meet three other fixed lines, the locus of the vertex opposite that face is a twisted cubic. The different positions of the face form a system of planes which divide homographically the three lines which meet its three edges, and thus the other three faces are corresponding planes of three homographic systems.*

339. From the theorems of the last article it follows, conversely, that "the *planes joining four fixed points of the system to any variable 'line through two points' form a constant anharmonic pencil*," and that "*four fixed planes of the system divide any 'line in two planes' in a constant anharmonic ratio.*"

It is very easy to prove these theorems independently from the anharmonic properties of conics. The first follows by projecting on to any fixed plane from one extremity of the 'line through two points,' the second reciprocally, since we know that the section of the developable by any plane A of the system,† consists of the corresponding line a of the system twice, together with a conic to which all other planes of the system are tangents. Thus, then, the anharmonic property of the tangents to a conic shows that four of these planes cut any two lines in two planes, AB, AC in the same anharmonic ratio; and, in like manner, AC is cut in the same ratio as CD.

As a particular case of these theorems, since the lines of the system are both lines in two planes and lines through two points; *four fixed planes of the system cut all the lines of the system in the same anharmonic ratio;* and *the planes joining four fixed points of the system to all the lines of the system are a constant anharmonic pencil.*

Many particular inferences may be drawn from these theorems, as at *Conics*, Art. 326, which see.

Thus consider four points α, β, γ, δ; and let us express that the planes joining them to the lines a, b, and $\alpha\beta$, cut the line $\gamma\delta$ homographically. Let the planes A, B meet $\gamma\delta$ in points t, t'. Let the planes joining the line a to β, and the line b to α meet $\gamma\delta$ in k, k'. Then we have

$$\{tk\gamma\delta\} = \{k't'\gamma\delta\} = \{kk'\gamma\delta\}.$$

If the points t, k' coincide, it follows from the first equation that the points k, t' coincide, and from the second that the points t, t', γ, δ are a harmonic

* For other geometrical constructions for the twisted cubic, see *The Twisted Cubic*, by P. W. Woods (Cambridge, 1913), Art. 26.

† It is often convenient to denote the planes of the system by capital letters, the corresponding lines by italics, and the corresponding points by Greek letters.

system. Thus we obtain Cremona's theorem, that if a series of chords meet the line of intersection of any plane A with the plane joining the corresponding point α to any line b of the system, then they will also meet the line of intersection of the plane B with the plane joining β to a; and will be cut harmonically where they meet these two lines and where they meet the curve.

The reader will have no difficulty in seeing when it will happen that one of these lines passes to infinity, in which case the other line becomes a diameter.

340. We have seen that the sections of the developable by the planes of the system are conics. The line of intersection of two planes of the system is a common tangent to the two corresponding conics. Thus *the planes touching two fixed conics, themselves having the line in which their planes intersect as a common tangent, are osculating planes of a twisted cubic.* [This was proved otherwise in Art. 333*a*.]

We may investigate the *locus of the centres of the conics in which planes of the system cut the developable, or more generally the locus of the poles with respect to these conics of the intersections of their planes with a fixed plane.* Since in every plane we can draw a "line in two planes" we may suppose that the fixed plane passes through the intersection of two planes of the system A, B.

Now consider the section by any other plane C; the traces on that plane of A and B are tangents to that section, and the pole of any line through their intersection lies on their cord of contact, that is to say, lies on the line joining the points where the lines of the system a, b meet C. But since all planes of the system cut the lines a, b homographically, the joining lines generate a hyperboloid of one sheet, of which a and b are generators. However then the plane be drawn through the line AB, the locus of poles is on this hyperboloid. But further, it is evident that the pole of any plane through the intersection of A, B lies in the plane which is the harmonic conjugate of that plane with respect to those tangent planes. The locus therefore which we seek is *a plane conic.* It appears also from the construction that since the poles when any plane $A + \lambda B$ is taken for the fixed plane,

lie on a conic in the plane $A - \lambda B$; conversely the locus when the latter is taken for fixed plane is a conic in the former plane.*

341. In conclusion, it is obvious enough that *cubics may be divided into four species according to the different sections of the curve by the plane at infinity.* Thus that plane may either meet the curve in three real points; in one real and two imaginary points; in one real and two coincident points, that is to say, a line of the system may be at infinity; or lastly, in three coincident points, that is to say a plane of the system may be altogether at infinity. These species have been called the *cubical hyperbola*, *cubical ellipse*, *cubical hyperbolic parabola*, and *cubical parabola.*

It is plain that when the curve has real points at infinity, it has branches proceeding to infinity, the lines of the system corresponding to the points at infinity being asymptotes to the curve. But when the line of the system is itself at in finity, as in the third and fourth cases, the branches of the curve are of a parabolic form proceeding to infinity without tending to approach to any finite asymptote. Since the quadric cones which contain the curve become cylinders when their vertices pass to infinity, it is plain that three quadric cylinders can be described containing the curve, the edges of the cylinders being parallel to the asymptotes. Of course in the case of the cubical ellipse two of these cylinders are imaginary: in the case of the hyperbolic parabola there are only two cylinders, one of which is parabolic, and in the case of the cubical parabola there is but one cylinder which is parabolic. The cubical ellipse may be conceived as lying on an elliptic cylinder, one generating line of which is the asymptote; the curve is a continuous line winding once round the cylinder, and approaching the asymptote on opposite sides at its two extremities.

It follows, from Art. 336, that in the case of the cubical

* The theorems of this article are taken from Cremona's paper.

ellipse the plane at infinity contains a real line in two planes, which is imaginary in the case of the cubical hyperbola. That is to say, in the former case, but not in the latter, two planes of the system can be parallel. From the anharmonic property we infer that in the case of the cubical parabola three planes of the system divide in a constant ratio all the lines of the system. In this case all the planes of the system cut the developable in parabolas. The system may be regarded as the envelope of $xt^3 - 3yt^2 + 3zt - d$ where d is constant. For further details we refer to Cremona's Memoir.

342. We proceed now to the *classification of curves of higher orders*. We have proved (Art. 331) that through any curve can be described two surfaces, the lowest values of whose degrees in each case there is no difficulty in determining. It is evident then, on the other hand, that if commencing with the simplest values of μ and ν we discuss all the different cases of the intersection of two surfaces whose degrees are μ and ν, we shall include all possible curves up to the r^{th} order, the value of this limit r being in each case easy to find when μ and ν are given. With a view to such a discussion we commence by investigating *the characteristics of the curve of intersection of two surfaces.** We have obviously $m = \mu\nu$, and if the surfaces are without multiple lines and do not touch, as we shall suppose they do not, their curve of intersection has no multiple points (Art. 203), and therefore $\beta = 0$. In order to determine completely the character of the system, it is necessary to know one more of its singularities, and we chose to seek for r, the degree of the developable generated by the tangents. Now this developable is got by eliminating $x'y'z'$ between the four equations

$$U' = 0,\ V' = 0,\ U_1'x + U_2'y + U_3'z + U_4'w = 0,$$
$$V_1'x + V_2'y + V_3'z + V_4'w = 0.$$

These equations are respectively of the degrees μ, ν, $\mu - 1$,

* The theory explained in the remainder of this section is taken from my paper dated July, 1849, *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 28.

$\nu - 1$: and since only the last two contain xyz, these variables enter into the result in the degree

$$\mu\nu\ (\nu - 1) + \mu\nu\ (\mu - 1) = \mu\nu\ (\mu + \nu - 2).$$

Otherwise thus: the condition that a line of the system should intersect the arbitrary line

$$\alpha x + \beta y + \gamma z + \delta w,\ \alpha' x + \beta' y + \gamma' z + \delta' w$$

is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \\ U_1, & U_2, & U_3, & U_4 \\ V_1, & V_2, & V_3, & V_4 \end{vmatrix} = 0,$$

which is evidently of the degree $\mu + \nu - 2$. This denotes a surface which is the locus of the points, the intersections of whose polar planes with respect to U and V meet the arbitrary line. And the points where this locus meets the curve UV are the points for which the tangents to that curve meet the arbitrary line.

Having then $m = \mu\nu$, $\beta = 0$, $r = \mu\nu\ (\mu + \nu - 2)$, we find, by Art. 327,

$$n = 3\mu\nu\ (\mu + \nu - 3),\ a = 2\mu\nu\ (3\mu + 3\nu - 10),\ 2h = \mu\nu\ (\mu - 1)\ (\nu - 1)$$

$$2g = \mu\nu\ \{\mu\nu\ (3\mu + 3\nu - 9)^2 - 22\ (\mu + \nu) + 71\},$$

$$2x = \mu\nu\ \{\mu\nu\ (\ \mu + \ \nu - 2)^2 - \ 4\ (\mu + \nu) + 8\},$$

$$2y = \mu\nu\ \{\mu\nu\ (\ \mu + \ \nu - 2)^2 - 10\ (\mu + \nu) + 28\}.$$

[Ex. Prove that the singularities H, G, v, ω (Art. 328) do not occur in the curve of intersection of two surfaces unless special conditions are fulfilled.]

343. We verify this result by determining independently h the number of "lines through two points" which can pass through a given point, that is to say, the number of lines which can be drawn through a given point so as to pass through two points of the intersection of U and V. For this purpose it is necessary to remind the reader of the method employed, Art. 121, Ex. 9, in order to find the equation of the cone whose vertex is any point and which passes through the intersection of U and V. Let us suppose that the vertex of the cone is taken on the curve, so as to have both U' and $V' = 0$

for the coordinates of the vertex $x'y'z'$. Then it appears that the equation of the cone is the result of eliminating λ between

$$\delta U' + \frac{\lambda}{1.2}\,\delta^2 U' + \frac{\lambda^2}{1.2.3}\,\delta^3 U' + \&c. = 0,$$

$$\delta V' + \frac{\lambda}{1.2}\,\delta^2 V' + \frac{\lambda^2}{1.2.3}\,\delta^3 V' + \&c. = 0.$$

These equations in λ are of the degrees $\mu - 1$, $\nu - 1$; $\delta U'$, $\delta^2 U'$, &c., contain the coordinates $x'y'z'$, xyz in the degrees $\mu - 1, 1$; $\mu - 2, 2$, &c. A specimen term of the result is $(\delta U')^{\nu-1}\, V^{\mu-1}$. Thus it appears that the result contains the variables xyz in the degree $\nu - 1 + \nu\,(\mu - 1) = \mu\nu - 1$; while it contains $x'y'z'$ in the degree $(\mu - 1)\,(\nu - 1)$. Every edge of this cone of the degree $\mu\nu - 1$, whose vertex is a point on the curve, is of course a "line through two points." If now in this result we consider the coordinates of any point xyz on the cone as known and $x'y'z'$ as sought, this equation of the degree $(\mu - 1)\,(\nu - 1)$ combined with the equations U and V determines the "points" belonging to all the "lines through two points" which can pass through the assumed point. The total number of such points is therefore $\mu\nu\,(\mu - 1)\,(\nu - 1)$, and the number of lines through two points is of course half this. The number of points thus determined has been called (Art. 325) the number of *apparent* double points on the intersection of the two surfaces.

344. Let us now consider the case when the curve UV has also *actual* double points; that is to say, *when the two surfaces touch* in one or more points. Now, in this case, the number of *apparent* double points remains precisely the same as in the last article, and the cone, standing on the curve of intersection and whose vertex is any point, has as double edges the lines joining the vertex to the points of contact in *addition*, to the number determined in the last article. It is easy to see that the investigation of the last article does not include the lines joining an arbitrary point to the points of contact. That investigation determines the number of cases when the radius vector from any point has two values

the same for both surfaces, but the radius vector to a point of contact has only one value the same for both, since the point of contact is not a double point on either surface Every point of contact then adds one to the number of double edges on the cone, and therefore diminishes the degree of the developable by two This might also be deduced from Art 342, since the surface generated by the tangents to the curve of intersection must include as a factor the tangent plane at a point of contact, since every tangent line in that plane touches the curve of intersection

If the surfaces have stationary contact at any point (Art 204) the line joining this point to the vertex of the cone is a cuspidal edge of that cone If the surfaces touch in t points of ordinary contact and in β of stationary contact, we have

$$m = \mu\nu,\ \beta = \beta,\ 2h = \mu\nu\ (\mu - 1)\ (\nu - 1),$$
$$r - \mu\nu\ (\mu + \nu - 2) - 2t - 3\beta,$$

and in applying Plucker s equations as in Arts 327, 342 we must put $\delta = h + t$ instead of $\delta = h$, the reader can calculate without difficulty how the other numbers in Art 342 are to be modified

We can hence obtain *a limit to the number of points at which two surfaces can touch if their intersection do not break up into curves of lower order*, for we have only to subtract the number of apparent double points from the maximum number of double points which a plane curve of the degree $\mu\nu$ can have (*Higher Plane Curves*, Art 42)

345 We shall now show that *when the curve of intersection of two surfaces breaks up into two simpler curves, the characteristics of these curves are so connected that, when those of the one are known, those of the other can be found.* It was proved (Art 343) that the points belonging to the "lines through two points' which pass through a given point are the intersection of the curve UV with a surface whose degree is $(\mu - 1)\ (\nu - 1)$ Suppose now that the curve of intersection breaks up into two whose degrees are m and m, where $m + m = \mu\nu$, then evidently the "two points" on any of these lines must either lie both on the curve m, both on the curve

m', or one on one curve and the other on the other. Let the number of lines through two points of the first curve be h, those for the second curve h', and let H be the number of lines which pass through a point on each curve, or, in other words, the number of *apparent intersections* of the curves. Considering then the points where each of the curves meets the surface of the degree $(\mu - 1)\ (\nu - 1)$, we have obviously the equations

$$m\ (\mu - 1)\ (\nu - 1) = 2h + H,\ m'(\mu - 1)\ (\nu - 1) = 2h' + H,$$

whence $\qquad 2\ (h - h') = (m - m')\ (\mu - 1)\ (\nu - 1).$

Thus when m and h are known m' and h' can be found. To take an example which we have already discussed, let the intersection of two quadrics consist in part of a right line (for which $m' = 1$, $h' = 0$), then the remaining intersection must be of the third degree $m = 3$, and the equation above written determines $h = 1$.

346. In like manner it was proved (Art. 342) that the locus of points, the intersection of whose polar planes with regard to U and V meets an arbitrary line, is a surface of the degree $\mu + \nu - 2$. The first curve meets this surface in the t points where the curves m and m' intersect (since U and V touch at these points) and in the r points for which the tangent to the curve meets the arbitrary line. Thus, then,

$$m\ (\mu + \nu - 2) = r + t,\ m'(\mu + \nu - 2) = r' + t,$$

$$(m - m')\ (\mu + \nu - 2) = r - r',$$

an equation which can easily be proved to follow from that in the last article.

The intersection of the cones which stand on the curves m, m' consists of the t lines to the points of actual meeting of the curves and of the H lines of apparent intersection; and the equation $H + t = mm'$ is easily verified by using the values just found for H and t, remembering also that $m' = \mu\nu - m$, $r = m\ (m - 1) - 2h$.

347. Having now established the principles which we shall have occasion to employ, we resume our enumeration of the different species of curves of the fourth order.

Every quartic curve lies on a quadric. For the quadric determined by nine points on the curve must altogether contain the curve (Art. 331). It is not generally true that a second quadric can be described through the curve; there are therefore *two principal families of quartics*, viz. *those which are the intersection of two quadrics, and those through which only one quadric can pass.**

We commence with the curves of the first family. The characteristics of the intersection of two quadrics which *do not touch* are (Art. 342)

$m=4,\ n=12,\ r=8,\ a=16,\ \beta=0,\ x=16,\ y=8,\ g=38,\ h=2.$

Several of these results can be established independently. Thus we have given (Art. 218) the equation of the developable generated by the tangents to the curve, which is of the eighth degree. It is there proved also that the developable has in each of its four principal planes a double line of the fourth order, whence $x=16$. It has been mentioned (Art. 216) that the developable circumscribing two quadrics has, as double lines, a conic in each of the principal planes. The number $y=8$ is thus accounted for. Again, it is shown (Art. 218), that the equation of the osculating plane is $Tu=T'v$ (u and v being the tangent planes to U and V at the point), which contains the coordinates of the point of contact in the third degree. If, then, it be required to draw an osculating plane through any assumed point, the points of contact are determined as the intersections of the curve UV with a surface of the third degree, the problem therefore admits of twelve solutions; thus $n=12$. Lastly, every generator of a quadric containing the curve is evidently a "line through two points" (Art. 345). Since, then, we can describe through any assumed point a quadric of the form $U+\lambda V$, the two generators of that quadric which pass through the point are two "lines through two points": or $h=2$. The lines through two points may be otherwise found by the following construction, the truth of

* The existence of this second family of quartics was first pointed out in the Memoir already referred to.

which it is easy to see: Draw a plane through the assumed point O, and through the intersection of its polar planes with respect to the two quadrics, this plane meets the quartic in four points which lie on two right lines intersecting in O.

A quartic of this species is determined by eight points (Art. 130).

[347*a*. *The coordinates of any point on a quartic curve of the first species may be expressed as elliptic functions of a parameter.* Two of the four cones of the system $U + \lambda V$ may be written

$$x^2 + y^2 - w^2 = 0,\ k^2x^2 + z^2 - w^2 = 0.$$

The first of these may be represented by $x = \sin\theta$, $y = \cos\theta$, $w = 1$. Hence these equations, with $z^2 = 1 - k^2\sin^2\theta$, represent the curve of intersection. If $\theta = \operatorname{am}(u, k)$ we get, in Jacobi's notation,

$$x = \operatorname{sn} u,\ y = \operatorname{cn} u,\ z = \operatorname{dn} u,\ w = 1.$$

Several of the properties of the curve can be determined from this form by the theory of elliptic functions. For example the parameters of the points where the plane $Ax + By + Cz + D = 0$ meets the curve are the four zeros of the function $A \operatorname{sn} u + B \operatorname{cn} u + C \operatorname{dn} u + D$. And if four points (u_1, u_2, u_3, u_4) are coplanar we have

$$u_1 + u_2 + u_3 + u_4 = 4mK + 4niK',$$

m and n being integers.

The coordinates may be expressed in Weierstrass's form $\wp u$ as follows.* By linear transformation of $U + \lambda V$ we can take for the two quadrics the cone $yw = z^2$, and the quadric $x^2 = cz^2 + dw^2 + 2fyz + 2nzw + 2myw$. Hence we may write

$$y = t^2,\ z = t,\ w = 1,\ x^2 = f(t)$$

where $f(t)$ is a cubic in t. Putting $s = \mu t + \nu$ we can reduce $f(t)$ to Weierstrass's type

$$4s^3 - g_2 s - g_3$$

and then putting $Y = \mu^2 y + 2\mu\nu z + \nu^2 w$, $Z = \mu z + \nu w$ we have

$$x = \wp'(u),\ Y = \wp^2(u),\ Z = \wp(u),\ w = 1.$$

W. Burnside † shows in another way that the elliptic differential involved is

$$\frac{d\lambda}{\sqrt{\lambda^4\Delta - \lambda^3\Theta + \lambda^2\Phi - \lambda\Theta' + \Delta'}},$$

g_2, g_3 being the invariants of the quartic in λ.

Ex. 1. Through the tangent line at any point on the quartic four planes can be drawn touching the quartic elsewhere, viz. the four tangent planes to the four quadric cones passing through the quartic.

The anharmonic ratio of these four planes is constant as the point moves along the curve.

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* See Bromwich, *Quadratic Forms*, p. 47.

† *Messenger of Mathematics*, XXIII. (1894), p. 89.

Ex. 2. Express the coordinates of the tangent plane to the developable formed by the quartic in terms of elliptic functions of a parameter.

Ex. 3. Prove that *the sixteen points of contact of the stationary planes of the quartic are the points where it meets the principal planes.*

A stationary plane is one containing four "consecutive" points on the curve, hence the points are given by $4u = 4mK + 4niK'$. It follows that u is either a zero of $\operatorname{sn} u$, of $\operatorname{cn} u$, or of $\operatorname{dn} u$, or it is an infinity of all three, therefore $x = 0$ or $y = 0$ or $z = 0$ or $w = 0$. For example $x = 0$ gives the four points $0, \pm 1, \pm 1, 1$.

This may also be proved by using the trigonometrical form; e.g. $\theta = 0$ will give a stationary plane if we apply the reciprocal of the method of Art. 324.

Ex. 4. The stationary planes corresponding to the four points in the plane $x = 0$ are $\pm k^2y \pm z + (1 + k^2)w = 0$ which are tangent planes to the cone $k^2y^2 - z^2 + w^2(1 - k^2)$ at the same points. Hence *the stationary planes are tangent planes, by fours, to the four cones* of the system $U + \lambda V$.

These results may be proved geometrically. The vertices of the four cones being A, B, C, D, a generator of the cone whose vertex is A meets the quartic in the two points P and Q, where it meets another of the cones, and if it meets the plane BCD in R, $APRQ$ is a harmonic range; also the tangent plane to the cone along the generator touches the quartic at both P and Q (cf. Ex. 1). Now if R lies on the quartic P, Q, R coincide and the tangent plane is stationary.

Ex. 5. Reciprocally, to find the 16 stationary points and the corresponding tangent planes, for the circum-developable of the system $\sigma + \lambda\sigma'$. Draw the four tangent lines from A to two of the double conics (Art. 216) whose planes meet in A, and join these tangent lines by pairs (one from each conic). These planes are the tangent planes at four stationary points, which are the points where the double conic in $x = 0$ is touched by the four generators of the developable lying in $x = 0$ (Art. 216).]

348. Secondly, let the two quadrics *touch*, then (Art. 344) the cone standing on the curve has a double edge more than in the former case, and the developable is of a degree less by two. Hence

$$m = 4,\ n = 6,\ r = 6\ ;\ g = 6,\ h = 3\ ;\ a = 4,\ \beta = 0\ ;\ x = 6,\ y = 4.$$

[It may be verified that the characteristics are the same as those of the reciprocal of the envelope of $at^4 + 4bt^3 + 6ct^2 + 4dt + e = 0$ (see Art. 329). But we shall see in the next paragraph that this property does not distinguish the quartic from those of the second family, which have the same characteristics. It may be shown however that *the coordinates of any point on the quartic considered may be expressed as rational quartic functions of a parameter.* For any two quadrics of the system $U + \lambda V = 0$, may in this case be written

$$ax^2 + by^2 + cz^2 + 2nzw = 0,$$
$$a'x^2 + b'y^2 + c'z^2 + 2n'zw = 0.$$

By eliminating zw and z^2 we may take for our two quadrics (the first being a cone)

$$x^2 + y^2 \quad - z^2 \; = 0$$
$$ax^2 + by^2 - zw = 0$$

and we may therefore write

$$x = \cos\theta,\ y = \sin\theta,\ z = 1,\ w = a\cos^2\theta + b\sin^2\theta:$$

putting $t = \tan\frac{1}{2}\theta$, the coordinates are expressed as quartic functions of t, and by linear transformation we can get

$$X = t^4,\ Y = 2t + 2t^3,\ Z = t^2,\ W = 1$$

which is the reciprocal of a system of the type

$$at^4 + 4bt^3 + 6ct^2 + 4\lambda bt + d = 0.]$$

Thirdly, the quadrics may have *stationary contact* at a point when we have

$$m=4,\ n=4,\ r=5;\ g=2,\ h=2;\ \alpha=1,\ \beta=1;\ x=2,\ y=2.$$

[To express the coordinates of this quartic rationally in terms of a parameter: If two quadrics touch their equations may be reduced to the forms

$$ax^2 + by^2 + 2fyz + 2zw = 0,\ a'x^2 + b'y^2 + 2f'yz + 2zw = 0,$$

and the condition for stationary contact (Art. 204) is either $a = a'$ or $b = b'$, but if $a = a'$ the quadrics do not intersect in a proper quartic. If $b = b'$, it will be seen that, after a simple transformation of coordinates, the equations of the cones of the system may be written $x^2 - yz = 0$, $y^2 - zw = 0$, the former representing three of the four cones which exist in general. The equations of the curve may thus be written

$$x = t,\ y = t^2,\ z = 1,\ w = t^4.]$$

This system, as noticed by Cayley, may be expressed as the envelope of

$$at^4 + 6ct^2 + 4dt + e,$$

where t is a variable parameter. The envelope is

$$(ae + 3c^2)^3 = 27(ace - ad^2 - c^3)^2,$$

which expanded contains a as a factor and so reduces to the fifth degree. The cuspidal edge is the intersection of $ae + 3c^2$, $4ce - 3d^2$. [Expressed in parameters it is $a = t^4$, $c = -t^2$, $d = 2t$, $e = -3$ which is a quartic of the type just mentioned.]

Since a cone of the fourth degree cannot have more than three double edges, two quadrics cannot touch in more points than one, unless their curve of intersection break up into simpler curves. If two quadrics *touch* at *two points* on the same generator, this right line is common to the surfaces, and the intersection breaks up into a right line and a cubic. If they touch at two points not on the same generator, the

intersection breaks up into two plane conics whose planes intersect in the line joining the points (see Art. 137).

349. If a quartic curve be *not the intersection of two quadrics* it must be the *partial intersection of a quadric and a cubic.* [Such a quartic is said to be of the second family.] We have already seen that the curve must lie on a quadric, and if through thirteen points on it, and six others which are not in the same plane,* we describe a cubic surface, it must contain the given curve. The intersection of this cubic with the quadric already found must be the given quartic together with a line of the second degree, and the apparent double points of the two curves are connected by the relation $h - h' = 2$, as appears on substituting in the formula of Art. 345 the values $m = 4$, $m' = 2$, $\mu = 3$, $\nu = 2$.

When the line of the second degree is a plane curve (whether conic or two right lines), we have $h' = 0$; therefore $h = 2$, or the quartic is one of the species already examined having two apparent double points. It is easy to see otherwise, that if a cubic and quadric have a plane curve common, through their remaining intersection a second quadric can be drawn; for the equations of the quadric and cubic are of the form $zw = u_2$, $zv_2 = u_2 x$, which intersect on $v_2 = xw$.

If, however, the cubic and quadric have common two right lines not in the same plane, this is a system having one apparent double point, since through any point can be drawn a transversal meeting both lines. Since then $h' = 1$, $h = 3$; or these quartics have three apparent double points, and are therefore essentially distinct from those already discussed which cannot have more than two. The numerical characteristics of these curves are precisely the same as those of the first species that are the intersections of two quadrics which touch (Art. 348), the cone standing on either curve having

* This limitation is necessary, otherwise the cubic might consist of the quadric and of a plane. Thus, if a curve of the fifth order lie in a quadric it cannot be proved that a cubic distinct from the quadric can contain the given curve; see *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 27.

three double edges, the difference being that one of the double edges in one case proceeds from an actual double point, while in the other they all proceed from apparent double points.

This system of quartics is the reciprocal of that given by the envelope of $at^4 + 4bt^3 + 6ct^2 + 4dt + e$. Moreover, this latter system has, in addition to its cuspidal curve of the sixth order, a nodal curve of the fourth, which is of the kind now treated of.

It is proved, as in Art. 335, that these quartics are met in three points by all the generators of the quadric on which they lie, which are of the same system as the lines common to the cubic and quadric; and are met once by the generators of the opposite system. The cone standing on the curve, whose vertex is any point of it, is then a cubic having a double edge, that double edge being one of the generators, passing through the vertex, of the quadric which contains the curve. Thus, while any cubic may be the projection of the intersection of two quadrics, quartics of this second family can only be projected into cubics having a double point. The quadric may be considered as the surface generated by all the "lines through three points" of the curve. It is plain, from what has been stated, that *every quartic, having three apparent double points, may be considered as the intersection of a quadric with a cone of the third order having one of the generators of the quadric as a double edge.*

[From the preceding theorem we can prove that *the coordinates of points on a quartic of the second species are quartic functions of a parameter*, and hence that its reciprocal is of the form mentioned.

A cubic cone whose vertex is 0, 0, 0, 1, having a double edge may be written $(x^2 - y^2)z = x^3$; therefore the coordinates of a section by $w = 0$ may be expressed as $x = 1 - \theta^2$, $y = \theta(1 - \theta^2)$, $z = 1$. A quadric having the double edge for a generator is of the form

$$ax^2 + 2hxy + by^2 + 2fyz + 2gzx - 2lxw - 2myw.$$

We may, if we please, choose w so that xw is a generator, and the other generator in w passes through xz. This gives $b = 0$, $f = 0$, $h = 0$. Hence

$$x = (l + m\theta)(1 - \theta^2), y = \theta(l + m\theta)(1 - \theta^2), z = l + m\theta, 2w = a(1 - \theta^2) + 2g.$$

Ex. 1. A quartic which is the intersection of two touching quadrics may be expressed in the form

$$x = t^4 + a, y = t^3, z = t^2, w = t;$$

and a quartic of the second species in the form

$$x = t^4 + a,\ y = t^3 + b,\ z = t^2 + c,\ w = t + d$$

where a, b, c, d are constants but if $a(c + d^2) = b^2 + 2bcd - c^3$, this reduces to a quartic of the first species.]*

350. Cayley has remarked that *it is possible to describe through eight points a quartic of this second family.* We want to describe through the eight points a cone of the third degree having its vertex at one of them, and having a double edge, which edge shall be a generator of a quadric through the eight points. Now it follows, from Art. 347, that if a system of quadrics be described through eight points, all the generators at any one of them lie on a cone of the third degree, which passes through the quartic curve of the first family determined by the eight points. Further, if S, S', S'' be three cubical cones having a common vertex and passing through seven other points, $\lambda S + \mu S' + \nu S''$ is the general equation of a cone fulfilling the same conditions; and if it have a double edge, $\lambda S_1 + \mu S_1' + \nu S_1''$ passes through that edge. Eliminating then λ, μ, ν between the three differentials, the locus of double edges is the cone of the sixth order

$$S_1(S_2'S_3'' - S_2''S_3') + S_2(S_3'S_1'' - S_3''S_1') + S_3(S_1'S_2'' - S_1''S_2') = 0.$$

The intersection then of this cone of the sixth degree with the other of the third determines right lines, through any of which can be described a quadric and a cubic cone fulfilling the given conditions. It is to be observed, however, that the lines, connecting the assumed vertex with the seven other points are simple edges on one of these cones and double edges on the other, and these (equivalent to fourteen intersections) are irrelevant to the solution of the problem. *Four quartics, therefore, can be described through the points.*

351. Prof. Cayley has directed my attention to a special case of this second family of quartics which I had omitted to notice. It is, when the curve has a *linear inflexion* of the kind

* [See Forsyth (*Quarterly Journal of Mathematics*, 27 (1895), "Twisted Quartics of the Second Species") who gives an analytical discussion of the modes of representing twisted cubics and quartics by parameters.]

noticed, Arts. 324, 328; that is to say, when three consecutive points of the curve are on a right line. Such a point obviously cannot exist on a quartic of the first family; for the line joining the three points must then be a generator of both quadrics, whose intersection would therefore break up into a line and a cubic, and would no longer be a quartic. Let us examine then in what case three consecutive planes of the system $at^4 + 4bt^3 + 6ct^2 + 4dt + e$ can pass through the same line. If such a case occurs, we may suppose that we have so transformed the equation that the singular point in question may answer to $t = \infty$; the three planes a, b, c, must therefore pass through the same line; or c must be of the form $\lambda a + \mu b$. But we may then transform the equation further by writing for t, $t + \theta$, and determining θ so that the quantity multiplying b in the coefficient of t^2 shall vanish. The *system then is the envelope of a plane*

$$at^4 + 4bt^3 + 6\lambda at^2 + 4dt + e.$$

A still more special case is when λ vanishes, or when the plane reduces to $at^4 + 4bt^3 + 4dt + e$; it is obvious then, that we have *two* points of linear inflexion; one answering to $t = \infty$, the other to $t = 0$. The developable in this latter case is

$$(ae - 4bd)^3 = 27\,(ad^2 + eb^2)^2;$$

which has for its edge of regression the intersection of $ae - 4bd$ with $ad^2 + eb^2$; but this consists of a curve of the fourth degree with the lines ab, de. *This system then is one whose reciprocal is of the same nature;* for we have $m = n = 4$, $h = g = 3$, $x = y = 4$. And the section of the developable by any plane has six cusps, viz. the four points where the plane meets the cuspidal edge, and the two where it meets the double generators ab, de. In the case previously noticed where c does not vanish but is equal to λa, there is but one point of linear inflexion; the envelope in question is, then, the reciprocal of a system for which $m = 4$, $n = 5$, $r = 6$, $h = 3$, $g = 4$, $x = 6$, $y = 4$. Another special case to be considered is when a curve has a double tangent; such a line being doubly a line

of the system is a double line on the developable. But this does not occur in quartic * curves.

To complete the enumeration of curves up to the fourth order, it would be necessary to classify, according to their apparent double points, improper systems made up of simpler curves of lower orders. Thus we have, for $m=2$, $h=1$, two lines not in the same plane; $m=3$, $h=1$, a conic and a line once meeting it; $h=2$, a conic and a line not meeting it; $h=3$, three lines, no two of which are in the same plane; $m=4$, $h=2$, a plane cubic and line once meeting it, or a twisted cubic and line twice meeting it, or two conics having two points common; $m=4$, $h=3$, a plane cubic and line not meeting it, or a twisted cubic and line once meeting it, or two conics having one point common: $m=4$, $h=4$, a twisted cubic and non-intersecting line, or two non-intersecting conics; $h=5$, a conic and two lines meeting neither the conic nor each other; $h=6$, four lines, no two of which are in the same plane.

An interesting quartic curve, Sylvester's *Twisted Cartesian* (see *Phil. Mag.*, 1866, pp. 287, 380), may here be mentioned specially: viz. *the locus of a point whose distances from three fixed foci are connected by the relations*

$$l\rho + m\rho' + n\rho'' = a,\ l'\rho + m'\rho' + n'\rho'' = b.$$

This curve has an infinity of foci lying in a plane cubic which is the locus of foci of conics which pass through four points lying on a circle; and may be represented as the intersection of a sphere and a parabolic cylinder.

352. The enumeration in regard to *curves of the fifth order* is effected in the memoir already cited. It is easy to see that besides plane quintics we have, I., quintics which are the partial intersection of a quadric and a cubic, the remaining intersection being a right line. These quintics have four apparent double points, and may besides have two actual nodal or cuspidal points. We may have, II., quintics with five apparent double points, and which may, besides, have one actual nodal or cuspidal point; these curves being the partial intersection of two cubics, and the remaining intersection a quartic of the second class. We may have, III., quintics with six apparent double points, being the partial intersection of two cubics, the remaining intersection being an improper quartic with four apparent double points. To these may be

* For other properties of curves of the fourth order, see papers by Chasles, *Comptes rendus*, Vols. LIV. and LV.; and by Cremona, *Memoirs of the Bologna Academy*, 1861. [For further references see Loria, *Il Passato ed il Presente delle Principali Teorie Geometriche* (1907), pp. 136-140, 382.]

added, IV., quintics with six apparent double points which are the partial intersection of a quadric and a quartic surface; the remaining intersection being three lines not in the same plane.

353. Instead of proceeding, as we have done, to enumerate the species of curves arranged according to their respective orders, we might have arranged our discussion according to the order r of the developables generated, and have enumerated the different species of the developables of the fourth, fifth, &c., orders. This is the method followed by Chasles, who has enumerated the species of developables up to the sixth order (*Comptes rendus*, Vol. LIV.), and by Schwarz (*Crelle*, Vol. LXIV., p. 1) who has carried on his enumeration to the seventh order. Schwarz's discussion contains the answer to the following question started by Cayley: the equation considered, Art. 329, where the parameter enters rationally, denotes a single plane whose envelope is a class of developables which Cayley calls *planar* developables; on the other hand, if the parameter entered by radicals, the equation rationalized would denote a system of planes whose envelope would therefore be called a *multiplanar* developable: now it is proposed to ascertain concerning each developable, what is, in this sense, the degree of its planarity. Schwarz has answered this question by showing that the *developables of the first seven orders are all planar.*

In fact when a developable is planar, the planes, lines and points of the system are expressible rationally by means of a parameter; and therefore every section of the developable is unicursal (*Higher Plane Curves*, Art. 44), as is also the cuspidal edge and every cone standing on it. It may be verified by the equations of Arts. 326-7, that

$$\tfrac{1}{2}(r-1)(r-2)-(m+x)=\tfrac{1}{2}(r-1)(r-2)-(n+y)=$$
$$\tfrac{1}{2}(m-1)(m-2)-(h+\beta)=\tfrac{1}{2}(n-1)(n-2)-(g+a)$$
$$=\tfrac{1}{2}(m+n)-(r-1),$$

any of these expressions denoting the deficiency either of the section (Art. 326) or of the cone (Art. 327). When this de-

ficiency vanishes, the developable is planar; when it $=1$ it is biplanar, &c. And this number is the same for a curve in space and for any other derived from it by linear transformation.

354. The discussion of the possible characteristics of a developable of given order, depends on the principle (Art. 330) that the section by a plane of the system is a curve of degree $r-2$ having $m-3$ cusps. Thus, if the developable be of the fifth order the section by a plane of the system is a cubic; and as this can have no more than one cusp, the edge of regression is at most of the fourth degree. And it cannot be of lower degree, since we have already seen that twisted cubics generate developables only of the fourth order. Hence the only developables * of the fifth order are those, considered Art. 348, generated by a curve of the fourth order.

In the same manner the section of a developable of the sixth order by a plane of the system is a quartic, which may have one, two, or three cusps. We have therefore $m=4$, 5, or 6; and, in like manner, n is confined within the same limits; and therefore (Art. 330) the section by the plane of the system is at most of the fifth class. Now a curve of the fourth degree with one cusp must have two other double points if it is only of the fifth class: and, if it have two cusps, it must have one other double point. In any case, therefore, this quartic is unicursal and the developable is planar. The case when the quartic has only one cusp (or $m=4$) has been already considered. The edge of regression has a nodal point; and the system is the reciprocal of the envelope of

$$at^4+4bt^3+6ct^2+4dt+\lambda a=0,$$

where there is a double plane of the system answering to $t=0$ and also to $t=\infty$.

If, again, the quartic section have three cusps, it is of the third class, and therefore for the developable $n=4$. This then

* The properties of these developables are treated of by Cremona, *Comptes rendus*, Vol. LIV. p. 604.

is also a case already discussed, Art. 349, the developable being the envelope of

$$at^4 + 4bt^3 + 6ct^2 + 4dt + e = 0.$$

Lastly, when the quartic has two cusps, it must, as we have seen, also have a double point, and therefore be of the fourth class. Hence $n = 5$. From the preceding formulæ the characteristics of a system for which $m = n = 5$, $r = 6$, are $g = h = 4$, $x = y = 5$, $\alpha = \beta = 2$; and, if we take the two stationary planes answering to $t = \infty$, $t = 0$, the system is the envelope of

$$at^5 + 5\lambda at^4 + 10ct^3 + 10dt^2 + 5\mu ft + f = 0.$$

Schwarz has noticed that the stationary tangent planes may be replaced by a triple tangent plane; that is to say, the system may be the envelope of

$$at^5 + 5\lambda at^4 + 10\mu at^3 + 10dt^2 + 5et + f = 0.$$

I have not examined with any care the theory of the effects of triple points of the curve of intersection of two surfaces on the number of its apparent double points. But (considering the case where λ and μ vanish in the equation last written) if we make b and $e = 0$ in the equations which I have given (*Cambridge and Dublin Mathematical Journal*, v. 158) for the edge of regression of the developable which results as the envelope of a quintic, the edge of regression is found to be the intersection of $2e^2 - 3df$ with $af^2 - 12d^2e$. And this intersection is the right line ef with a curve of the fifth order, having the point def for a triple point. For this being a double point on each surface is a quadruple point on their curve of intersection; and since the right line passes through the point def, the remaining curve has a triple point at that point.

355. We shall conclude this section by applying some of the results already obtained in it, to the solution of a problem which occasionally presents itself: "*Three surfaces whose degrees are μ, ν, ρ, have a certain curve common to all three; how many of their $\mu\nu\rho$ points of intersection are absorbed by the curve?* In other words, in how many points do the

surfaces intersect in addition to this common curve?" Now let the first two surfaces intersect in the given curve, whose degree is m, and in a complementary curve $\mu\nu - m$, then the points of intersection not on the first curve must be included in the $(\mu\nu - m)\rho$ intersections of the latter curve with the third surface. But some of these intersections are on the curve m, since it was proved (Art. 346) that the latter curve intersects the complementary curve in $m(\mu+\nu-2)-r$ points. Deducting this number from $(\mu\nu - m)\rho$ we find that the surfaces intersect in $\mu\nu\rho - m(\mu+\nu+\rho-2)+r$ points which are not on the curve m; or that *the common curve absorbs* $m(\mu+\nu+\rho-2)-r$ *points of intersection.*

Ex. Applying this formula to the intersections of three cubics having a common curve of degree m, the number of residual points not on the curve m is $27 - 7m + r$. Now supposing the surfaces have four right lines common, this at first seems to give $m = 4$, $h = 6$, hence $r = 0$ and the number of residual points -1. But it is easily seen that the cubic surfaces in this case have also common the two transversals of the four right lines, and these have also an apparent double point; hence, the values should have been taken $m = 6$ $h = 7$, and these give the number of remaining points of intersection $= 1$.

If the common curve be two conics, the line in which their planes intersect is also contained in the surfaces and thus $m = 5$, $h = 4$ give 4 remaining intersections.

In precisely the same way we solve the corresponding question *if the common curve be a double curve on the surface* ρ. We have then to subtract from the number $(\mu\nu - m)\rho$, $2\{m(\mu+\nu-2)-r\}$ points, and we find that the common curve diminishes the intersections by $m(\rho+2\mu+2\nu-4)-2r$ points.

These numbers, expressed in terms of the apparent double points of the curve m, are

$$m(\mu+\nu+\rho-m-1)+2h \text{ and } m(\rho+2\mu+2\nu-2m-2)+4h.$$

356. The last article enables us to answer the question: "*If the intersection of two surfaces is in part a curve of degree m, which is a double curve on one of the surfaces, in how many points does it meet the complementary curve of intersection?*" Thus, in the question last considered, the

surfaces μ, ρ intersect in a double curve m and a complementary curve $\mu\rho - 2m$; and the points of intersection of the three surfaces are got by subtracting from $(\mu\rho - 2m)\,\nu$ the number of intersections of the double curve with the complementary. Hence

$$(\mu\rho - 2m)\,\nu - \iota = \mu\nu\rho - m\,(\rho + 2\mu + 2\nu - 4) + 2r,$$

whence

$$\iota = m\,(\rho + 2\mu - 4) - 2r.$$

We can verify this formula when the curve m is the complete intersection of two surfaces U, V, whose degrees are k and l. Then ρ is of the form $AU^2 + BUV + CV^2$ where A is of the degree $\rho - 2k$, &c., and μ is of the form $DU + EV$ where D is of the degree $\mu - k$. The intersections of the double curve with the complementary are the points for which one of the tangent planes to one surface at a point on the double curve coincides with the tangent plane to the other surface. They are therefore the intersections of the curve UV with the surface $AE^2 - BDE + CD^2$ which is of the degree $\rho + 2\mu - 2(k + l)$. The number of intersections is $kl\,\{\rho + 2\mu - 2\,(k + l)\}$ which coincides with the formula already obtained on putting $kl = m$, $kl\,(k + l - 2) = r$.

357. From the preceding article we can show how, *when two surfaces partially intersect in a curve which is a double curve on one of them, the singularities of this curve and its complementary are connected.* The first equation of Art. 346 ceases to be applicable because the surface $\mu + \rho - 2$ altogether contains the double curve, but the second equation gives us

$$m'\,(\mu + \rho - 2) = 2\iota + r' = r' + 2m\,(\mu + 2\rho - 4) - 4r,$$

whence

$$4r - r' = (2m - m')\,(\mu + \rho - 2) + 2m\,(\rho - 2).$$

In like manner we find that the apparent double points of the two curves are connected by the relation

$$8h - 2h' = (2m - m')\,(\mu - 1)\,(\rho - 1) - 2m\,(\rho - 1).$$

Ex. When a quadric passes through a double line on a cubic the remaining intersection is of the fourth degree, of the sixth rank, and has three apparent double points.

Section III. Non-projective Properties of Curves.

358. As we shall more than once in this section have occasion to consider lines indefinitely close to each other, it is convenient to commence by showing how some of the formulæ obtained in the first chapter are modified when the

lines considered are indefinitely near. We proved (Art. 14) that the angle of inclination of two lines is given by the formula

$$\sin^2\theta = (\cos\beta\cos\gamma' - \cos\beta'\cos\gamma)^2 + (\cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha)^2 + (\cos\alpha\cos\beta' - \cos\alpha'\cos\beta)^2.$$

When the lines are indefinitely near we may substitute for $\cos\alpha'$, $\cos\alpha + \delta\cos\alpha$, &c., and put $\sin\theta = \delta\theta$, when we have

$$\delta\theta^2 = (\cos\beta\,\delta\cos\gamma - \cos\gamma\,\delta\cos\beta)^2 + (\cos\gamma\,\delta\cos\alpha - \cos\alpha\,\delta\cos\gamma)^2 + (\cos\alpha\,\delta\cos\beta - \cos\beta\,\delta\cos\alpha)^2.$$

If the direction-cosines of any line be $\frac{l}{r}$, $\frac{m}{r}$, $\frac{n}{r}$ where $l^2 + m^2 + n^2 = r^2$, the preceding formula gives

$$r^4\delta\theta^2 = (m\delta n - n\delta m)^2 + (n\delta l - l\delta n)^2 + (l\delta m - m\delta l)^2.$$

Since we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

$$\cos\alpha\,\delta\cos\alpha + \cos\beta\,\delta\cos\beta + \cos\gamma\,\delta\cos\gamma = 0;$$

if we square the latter equation and add it to the expression for $\delta\theta^2$, we get another useful form

$$\delta\theta^2 = (\delta\cos\alpha)^2 + (\delta\cos\beta)^2 + (\delta\cos\gamma)^2.$$

It was proved (Art. 15) that $\cos\beta\cos\gamma' - \cos\beta'\cos\gamma$, &c. are proportional to the direction-cosines of the perpendicular to the plane of the two lines. It follows then, that the direction-cosines of the perpendicular to the plane of the consecutive lines just considered are proportional to $m\delta n - n\delta m$, $n\delta l - l\delta n$, $l\delta m - m\delta l$, the common divisor being $r^2\delta\theta$.

Again, it was proved (Art. 46) that the direction-cosines of the line bisecting the external angle made with each other by two lines are proportional to

$$\cos\alpha - \cos\alpha',\ \cos\beta - \cos\beta',\ \cos\gamma - \cos\gamma',\ \&c.$$

Hence, when two lines are indefinitely near, the direction-cosines of a line drawn in their plane, and perpendicular to their common direction are proportional to $\delta\cos\alpha$, $\delta\cos\beta$, $\delta\cos\gamma$, the common divisor being $\delta\theta$.

[If lines be drawn through the centre of a unit sphere in the positive direction of the tangents to the curve, they meet the sphere in a curve which is sometimes called the *spherical indicatrix* of the original curve. The formulæ of this article follow simply from considering this spherical curve.

Ex. If the spherical indicatrix is $\xi = \phi(\theta)$, $\eta = \psi(\theta)$, $\zeta = \chi(\theta)$, where $\phi^2 + \psi^2 + \chi^2 = 1$, the curves which it represents are $x = a + \int\lambda\phi d\theta$, $y = b + \int\lambda\psi d\theta$, $z = c + \int\lambda\chi d\theta$, where λ is an arbitrary function of θ. And if the original curve is $x = \phi(\theta)$, $y = \psi(\theta)$, $z = \chi(\theta)$ the spherical indicatrix is

$$\xi = \frac{\phi'(\theta)}{\sqrt{\phi'^2 + \psi'^2 + \chi'^2}} \text{ \&c.]}$$

359. We proved (Art. 317) that the direction-cosines of a tangent to a curve are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, while, if the curve be given as the intersection of two surfaces, these cosines are proportional to $MN' - M'N$, $NL' - N'L$, $LM' - L'M$, where L, M, &c. denote the first differential coefficients.

An infinity of normal lines can evidently be drawn at any point of the curve. Of these, two have been distinguished by special names; the normal which lies in the osculating plane is commonly called the *principal normal;* and the normal perpendicular to that plane, being normal to two consecutive elements of the curve, has been called by M. Saint-Venant the *binormal.* At any point of the curve, the tangent, the principal normal, and the binormal form a system of three rectangular axes.

All the normals lie in the *normal plane,* the plane perpendicular to the tangent line, viz.

$$(x - x')\,dx + (y - y')\,dy + (z - z')\,dz = 0$$

in the one notation; or in the other

$$(MN' - M'N)(x - x') + (NL' - N'L)(y - y') + (LM' - L'M)(z - z') = 0.$$

360. Let us consider now the *equation of the osculating plane.** Since it contains two consecutive tangents of the curve, its direction-cosines (Art. 358), which are those of the *binormal,* are proportional to

$$dyd^2z - dzd^2y,\; dzd^2x - dxd^2z,\; dxd^2y - dyd^2x,$$

quantities which, for brevity, we shall call X, Y, Z. The equation of the osculating plane is therefore

$$X(x - x') + Y(y - y') + Z(z - z') = 0.$$

* [The osculating plane as well as the tangent line are *projective* elements of curves.]

The same equation might have been obtained (by Art. 31) by forming the equation of the plane joining the three consecutive points

$$x', y', z'; \; x' + dx', \; y' + dy', \; z' + dz';$$
$$x' + 2dx' + d^2x', \; y' + 2dy' + d^2y', \; z' + 2dz' + d^2z'.$$

In applying this formula, we may simplify it by taking one of the coordinates at pleasure as the independent variable, and so making d^2x', d^2y', or $d^2z' = 0$.

[Any plane through a tangent line has contact of the first order with the curve at the point; and therefore it satisfies the two equations

$$\lambda x' + \mu y' + \nu z' + \rho w' = 0, \; \lambda dx' + \mu dy' + \nu dz' + \rho dw' = 0.$$

An osculating plane has contact of the second order; hence we have also

$$\lambda d^2x' + \mu d^2y' + \nu d^2z' + \rho d^2w' = 0.$$

Therefore its equation in homogeneous coordinates can at once be written as a determinant.

A plane has contact of the n^{th} order if $\lambda d^rx' + \mu d^ry' + \nu d^rz' + \rho d^rw' = 0$ where $r = 0, 1, 2 \ldots n$. A *stationary* plane is one having contact of the third order (four "consecutive" points).]

361. In order to be able to illustrate by an example the application of the formulæ of this section, it is convenient here to form the equations and state some of the properties of the *circular helix* or curve formed by the thread of a screw. The *circular helix* may be defined as *the form assumed by a right line traced in any plane when that plane is wrapped round the surface of a right cylinder.** From this definition the equations of the helix are easily obtained. The equation of any right line $y = mx$ expresses that the ordinate is proportional to the intercept which that ordinate makes on the axis of x. If now the plane of the right line be wrapped round a right cylinder, so that the axis of x may coincide with the circular base, the right line will become a helix, and the ordinate of any point of the curve will be proportional to the intercept measured along the circle, which that ordinate

* Conversely, a circular helix becomes a right line when the cylinder on which it is traced is developed into a plane; and is, therefore, a geodesic on the cylinder (Art. 308). [A helix in general is a geodesic on a cylinder, or what is the same thing, the tangent line makes a constant angle with the axis.]

makes on the circular base, counting from the point where the helix cuts the base. Thus the coordinates of the projection on the plane of the base of any point of the helix are of the form $x = a \cos \theta, y = a \sin \theta$, where a is the radius of the circular base. But the height z has been just proved to be proportional to the arc θ. Hence, the equations of the helix are

$$x = a \cos \frac{z}{h},\ y = a \sin \frac{z}{h}, \text{ whence also } x^2 + y^2 = a^2,$$

or we may write $x = a \cos \theta$, $y = a \sin \theta$, $z = h\theta$ and treat θ as the independent variable.

We plainly get the same values for x and y when the arc increases by 2π, or when z increases by $2\pi h$; hence the interval between the threads of the screw is $2\pi h$.

Since we have

$$dx = -\frac{a}{h} \sin \frac{z}{h} dz = -\frac{y}{h} dz,\ dy = \frac{a}{h} \cos \frac{z}{h} dz = \frac{x}{h} dz,$$

we have $ds^2 = \dfrac{a^2 + h^2}{h^2} dz^2$. It follows that $\dfrac{dz}{ds}$ is constant, or the angle made by the tangent to the helix with the axis of z (which is the direction of the generators of the cylinder) is constant. It is easy to see that this is the same as the angle made with the generators by the line into which the helix is developed when the cylinder is developed into a plane.

The length of the arc of the curve is evidently in a constant ratio to the height ascended.

The equations of the tangent are (Art. 317)

$$\frac{x - x'}{y'} = -\frac{y - y'}{x'} = -\frac{z - z'}{h}.$$

If then x and y be the coordinates of the point where the tangent pierces the plane of the base, we have from the preceding equations

$$(x - x')^2 + (y - y')^2 = (x'^2 + y'^2) \frac{z'^2}{h^2} = a^2 \frac{z'^2}{h^2},$$

or the distance between the foot of the tangent and the projection of the point of contact is equal to the arc which

measures the distance along the circle of that projection from the initial point.

This also can be proved geometrically, for if we imagine the cylinder developed out on the tangent plane, the helix will coincide with the tangent line, and the line joining the foot of the tangent to the projection of the point of contact will be the arc of the circle developed into a right line. Thus, then, the locus of the point where the tangent meets the base is the involute of the circle.

The equation of the normal plane is

$$y'x - x'y = h\ (z - z').$$

To find the equation of the osculating plane we have

$$d^2x = -\frac{1}{h^2}xdz^2,\ d^2y = -\frac{1}{h^2}ydz^2,\ d^2z = 0,$$

whence the equation of the osculating plane is

$$h(y'x - x'y) + a^2(z - z') = 0.$$

The form of the equation shows that the osculating plane makes a constant angle with the plane of the base.

[Ex. 1. The coordinates of a point on any twisted cubic may be written

$$x = \theta^3,\ y = \theta^2,\ z = \theta,\ w = 1$$

(Art. 333*a*).

Find the osculating plane at any point.

Ex. 2. Prove that *three osculating planes can be drawn to the cubic through a point* (*P*) *not on the curve.*

Ex. 3. Prove that *the points of contact of these three planes lie in a plane* (*L*) *through the point.*

Ex. 4. The point P is called the focus of the plane L; prove that if A lies in a plane whose focus is B, then B lies in a plane whose focus is A.

Ex. 5. Find the tangent, normal plane and osculating plane at any point on the intersection of two coaxal quadrics.

The coordinates of a point on the curve may be expressed in the form (cf. Art. 347*a*)

$$x = A\cos\theta,\ y = B\sin\theta,\ z^2 = C\cos^2\theta + D\sin^2\theta,$$

but the method of the following article is more symmetrical.

Ex. 6. Prove that *the osculating plane of any curve at a non-singular point crosses the curve thereat.*

Let $\theta = 0$ be the origin and suppose x, y, z can be expressed in series converging at the point in the form

$$x = a_1\theta + a_2\theta^2 + a_3\theta^3 + \&c.,\ y = b_1\theta + \&c.,\ z = c_1\theta + \&c.$$

If we substitute the values of the coordinates of the points $\theta = h$, $\theta = -h$, in the equation of the osculating plane at $\theta = 0$, the results, excluding higher

powers of h are of the form Ah^3 and $-Ah^3$. Hence the two points are on opposite sides of the plane.

A plane having contact of odd order does not cross the curve; a plane with contact of even order crosses it.

Ex. 7. X is any point on the tangent line at P to a twisted curve and Y is any point on the binormal. Prove that the cone with vertex X standing on the curve, has PX for a cuspidal edge, the osculating plane PXY being the tangent plane along the edge.

If a cone is drawn through the curve with its vertex at Y, the plane PXY osculates the cone along PY.

Thus a curve seen from any point in the tangent line at P has the appearance of a curve with a cusp at P. The developable generated by tangent lines is therefore the locus of points from which the curve has *apparent* cusps (real cusps not being counted). The curve seen from any point on the osculating plane other than a point on the tangent line appears to have an inflexion at P.

Ex. 8 A rational curve being expressed by $x = \phi(\theta)$, $y = \psi(\theta)$, &c., ϕ, ψ, being integral functions of degree n, find the number of osculating planes through an arbitrary point not on the curve. Ans. $3(n-2)$.]

362. We can give the *equation of the osculating plane* a form more convenient in practice *when the curve is defined as the intersection of two surfaces U, V* Since the osculating plane passes through the tangent line. its equation must be of the form

$$\lambda\,(Lx + My + Nz + Pw) = \mu\,(L'x + M'y + N'z + P'w),$$

where $Lx +$ &c. is the tangent plane to the first surface, $L'x +$ &c. to the second. This equation is identically satisfied by the coordinates of a point common to the two surfaces, and by those of a consecutive point; and, on substituting the coordinates of a second consecutive point, we get

$$\mu = Ld^2x + Md^2y + Nd^2z + Pd^2w,\ \lambda = L'd^2x + M'd^2y + N'd^2z + P'd^2w.$$

But differentiating the equation

$$Ldx + Mdy + Ndz + Pdw = 0,$$

we get
$$Ld^2x + Md^2y + Nd^2z + Pd^2w = -U',$$

where $U' = adx^2 + bdy^2 + cdz^2 + ddw^2$
$+ 2fdydz + 2gdzdx + 2hdxdy + 2ldxdw + 2mdydw + 2ndzdw$,
where a, b, &c. are the second differential coefficients. Now dx, &c. satisfy the equations

$$Ldx + My + Ndz + Pdw = 0, \quad L'dx + M'dy + N'dz + P'dw = 0;$$

and since we may either, as in ordinary Cartesian equations, take w as constant; or else x, or y, or z; or, more generally, must take some linear function of these coordinates as constant; we may therefore combine with the two preceding equations the arbitrary equation

$$\alpha dx + \beta dy + \gamma dz + \delta dw = 0.$$

Now it can easily be verified that if we substitute in the equation of any quadric, the coordinates of the intersection of three planes

$$Lx + My + Nz + Pw,\quad L'x + M'y + N'z + P'w,\quad \alpha x + \beta y + \gamma z + \delta w,$$

the result U' will be proportional to the determinant (cf. Art. 80).

$$\begin{vmatrix} a, & h, & g, & l, & L, & L', & \alpha \\ h, & b, & f, & m, & M, & M', & \beta \\ g, & f, & c, & n, & N, & N', & \gamma \\ l, & m, & n, & d, & P, & P', & \delta \\ L, & M, & N, & P & & & \\ L', & M', & N', & P' & & & \\ \alpha, & \beta, & \gamma, & \delta & & & \end{vmatrix}$$

This determinant may be reduced by subtracting from the fifth column multiplied by $(m-1)$ the sum of the first four columns, multiplied respectively by x, y, z, w; when the whole of the fifth column vanishes, except the last row, which becomes $-(\alpha x + \beta y + \gamma z + \delta w)$. In like manner we may then subtract from the fifth row, multiplied by $(m-1)$, the sum of the first four rows multiplied respectively by x, y, z, w, when, in like manner, the whole of the fifth row vanishes, except the last column, which is $-(\alpha x + \beta y + \gamma z + \delta w)$. Thus the determinant reduces to

$$-\frac{(\alpha x + \beta y + \gamma z + \delta w)^2}{(m-1)^2} \begin{vmatrix} a, & h, & g, & l, & L' \\ h, & b, & f, & m, & M' \\ g, & f, & c, & n, & N' \\ l, & m, & n, & d, & P' \\ L', & M', & N', & P', & \end{vmatrix}.$$

If we call the determinant last written S, and the correspond-

ing determinant for the other equation S', the equation of the osculating plane is

$$\frac{S'}{(n-1)^2}(Lx+My+Nz+Pw)=\frac{S}{(m-1)^2}(L'x+M'y+N'z+P'w).*$$

This equation has been verified in the case of two quadrics, see Art. 218.

Ex. 1. To find the osculating plane of

$$ax^2+by^2+cz^2+dw^2,\ a'x^2+b'y^2+c'z^2+d'w^2.$$

Ans. $(ab'-ba')(ac'-ca')(ad'-da')\,x'^3x+(ba'-b'a)(bc'-b'c)(bd'-b'd)\,y'^3y$
$+(ca'-c'a)(cb'-c'b)(cd'-c'd)\,z'^3z+(da'-d'a)(db'-d'b)(dc'-d'c)\,w'^3w=0.$

Ex. 2. To find the osculating plane of the line of curvature

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1.$$

$$\textit{Ans.}\ \frac{a''^2xx'}{a^2a'^2}+\frac{b''^2yy'}{b^2b'^2}+\frac{c''^2zz}{c^2c'^2}=1.$$

363. The condition that four consecutive points should lie in one plane, or, in other words, that a point on the curve should be the point of contact of a *stationary plane*, is got by substituting in the equation of the plane through three consecutive points, the coordinates of a fourth consecutive point. Thus, from the equation of Art. 31, the condition required is the determinant

$$d^3x(dyd^2z-dzd^2y)+d^3y(dzd^2x-dxd^2z)+d^3z(dxd^2y-dyd^2x)=0.$$

If a curve in space be a plane curve, this condition must be fulfilled by the coordinates of every point of it.

For *a curve given as the intersection of two surfaces U, V*, Clebsch determined as follows (see *Crelle*, LXIII. 1) the *condition for a point of osculation.* Writing for brevity $S=(m-1)^2T$, $S'=(n-1)^2T'$, the equation given in the last article for the osculating plane is

$$(T'L-TL')x+(T'M-TM')y+(T'N-TN')z+(T'P-TP')w=0,$$

and the equation of a consecutive osculating plane differs from this by terms

$$(T'dL+LdT'-TdL'-L'dT)\ x+\&c.=0.$$

Thus, in order that the two planes may coincide, intro-

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* This equation is due to Hesse, see Crelle's *Journal*, Vol. XLI.

ducing an arbitrary differential dt, we must have the four equations

$$T'dL + LdT' - TdL' - L'dT = (T'L - TL')dt, \ \&c$$

If, now, we write

$$T = AL' + BM' + CN' + DP', \ T' = A'L + B'M + C'N + D'P,$$

where A, B, &c. are proportional to minors of the determinant S, and where in fact

$$A = \tfrac{1}{2}\frac{dT}{dL'}, \ B = \tfrac{1}{2}\frac{dT}{dM'}, \ \&c.,$$

we must have

$$AL + BM + CN + DP = 0, \ AdL + BdM + CdN + DdP = 0,$$
$$A'L' + \&c. = 0, \ A'dL' + \&c. = 0\,;$$

for, if in the determinant S we substitute for the last column either L, M, N, P, or dL, dM, dN, dP, it is easy to see that the determinant vanishes. Multiply then the four equations written above by A, B, C, D respectively, and add, and we have, after dividing by T,

$$dT + \tfrac{1}{2}\left(\frac{dT}{dL'}\,dL' + \frac{dT}{dM'}\,dM' + \frac{dT}{dN'}\,dN' + \frac{dT}{dP'}\,dP'\right) = Tdt,$$

which we may write

$$dT + \tfrac{1}{2}d(T) = Tdt,$$

where by $d(T)$ we mean the differential of T considered merely as a function of L', M', N', P'; a, b, &c. being regarded as constants. Similarly we have $dT' + \tfrac{1}{2}d(T') = T'dt$. Let us now write at full length for dT, $T_1dx + T_2dy +$ &c.; and eliminating dx, dy, dz, dw, dt between the two equations just obtained, and the three conditions which connect dx, dy, dz, dw, we obtain the required condition in the form of a determinant

$$\begin{vmatrix} T_1 + \tfrac{1}{2}(T_1), & T_2 + \tfrac{1}{2}(T_2), & T_3 + \tfrac{1}{2}(T_3), & T_4 + \tfrac{1}{2}(T_4), & T \\ T_1' + \tfrac{1}{2}(T_1'), & T_2' + \tfrac{1}{2}(T_2'), & T_3' + \tfrac{1}{2}(T_3'), & T_4' + \tfrac{1}{2}(T_4'), & T' \\ L, & M, & N, & P, & 0 \\ L', & M', & N', & P', & 0 \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix} = 0.$$

Now T is a function of x, y, z, w of the degree $3m + 2n - 8$, but when regard is paid only to the $xyzw$, which enter into

L', M', &c., (T) is of the degree $2(n-1)$; if, therefore, we multiply the first four columns by x, y, z, w respectively, and subtract them from $3(m+n-3)$ times the last column, the first four terms of the last column vanish, and the equation just written may be reduced by cancelling the fifth row and column of the determinant. The condition that we have just obtained is of the degree $6m+6n-20$ in the variables as might be inferred from the value of a, Art. 342.

If the surfaces U and V are quadrics, and therefore the coefficients a, b, &c. really constant, (T_1), (T_2), &c. are identical with T_1, T_2, &c., and the condition that we have obtained is the result of equating to zero the Jacobian of the four surfaces T, T', U, V.

[Ex. Prove that the points where the curve of intersection of two quadrics meet the principal planes are points of contact of stationary tangent planes and find the equations of these planes. Cf. Exs. 3 and 4, Art. 347.]

364. We shall next consider the *circle determined by three consecutive points* of the curve, which, as in plane curves, is called the circle of curvature. It obviously lies in the osculating plane: its centre is the intersection of the traces on that plane, by two consecutive normal planes; and its radius is commonly called the radius of *absolute* curvature, to distinguish it from the radius of *spherical* curvature, which is the radius of the sphere determined by four consecutive points on the curve, and which will be investigated presently. If through the centre of a circle a line be drawn perpendicular to its plane, any point on this line is equidistant from all the points of the circle, and may be called a pole of the circle. Now the intersection of two consecutive normal planes evidently passes through the centre of the circle of curvature, and is perpendicular to its plane. Monge has therefore called the lines of intersection of pairs of consecutive normal planes the *polar* lines of the curve. It is evident that *all the normal planes envelope a developable of which these polar lines are the generators*, and which accordingly has been called the *polar developable* surface. We shall presently state some

properties of this surface. The polar line is evidently parallel to the line called the binormal (Art. 359).

365. In order *to obtain the radius of curvature*, we shall first calculate the *angle of contact*, that is to say, the angle made with each other by two consecutive tangents to the curve. The direction-cosines of the tangent being $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, it follows, from Art. 358, that $d\theta$, the angle between two consecutive tangents, is given by either of the formulæ

$$d\theta^2 = \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2,$$

or
$$ds^4 d\theta^2 = X^2 + Y^2 + Z^2,$$

where
$$X = dyd^2z - dzd^2y, \text{ \&c.}$$

The truth of the latter formula may be seen geometrically; for the right-hand side of the equation denotes the square of double the triangle formed by three consecutive points (Art. 32); but two sides of this triangle are each ds, and the angle between them is $d\theta$, hence double the area is $ds^2 d\theta$.

If now ds be the element of the arc, the tangents at the extremities of which make with each other the angle $d\theta$, then since the angle made with each other by two tangents to a circle is equal to the angle that their points of contact subtend at its centre, we have $\rho d\theta = ds$. And the element of the arc and the two tangents being common to the curve and the circle of curvature, the radius of curvature is given by the formula

$$\rho = \frac{ds}{d\theta}; \text{ whence } \rho^2 = \frac{ds^2}{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2};$$

or
$$\rho^2 = \frac{ds^6}{X^2 + Y^2 + Z^2}.$$

By performing the differentiations indicated, another value for $d\theta$ is found without difficulty,

$$ds^2 d\theta^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2.*$$

* This formula may also be proved geometrically. Let AB, BC be two consecutive elements of the curve; AD a line parallel and equal to BC; then

Thus if s is the independent variable

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

Ex. To find the radius of curvature of the circular helix. Using the formulæ of Art. 361, we find $\rho = \frac{a^2 + h^2}{a}$; or the radius of curvature is constant.

366. Having thus determined the magnitude of the radius of curvature, we are enabled by the formulæ of Art. 358 also to determine its position. For the direction-cosines of a line drawn in the plane of two consecutive tangents, and perpendicular to their common direction, are, by that article,

$$\frac{1}{d\theta}d\frac{dx}{ds},\ \frac{1}{d\theta}d\frac{dy}{ds},\ \frac{1}{d\theta}d\frac{dz}{ds};\ \text{or } \rho\frac{d\frac{dx}{ds}}{ds},\ \rho\frac{d\frac{dy}{ds}}{ds},\ \rho\frac{d\frac{dz}{ds}}{ds}.$$

If x, y, z be the coordinates of a point on the curve, and x', y', z' those of the centre of curvature, then the projections of the radius of curvature on the axes are $x' - x$, $y' - y$, $z' - z$; but they are also $\rho\cos\alpha$, $\rho\cos\beta$, $\rho\cos\gamma$. Putting in then for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ their values just found, *the coordinates of the centre of curvature* are determined by the equations

$$x' - x = \rho^2\frac{d\frac{dx}{ds}}{ds},\ y' - y = \rho^2\frac{d\frac{dy}{ds}}{ds},\ z' - z = \rho^2\frac{d\frac{dz}{ds}}{ds}.$$

[Ex. 1. Find the radius and centre of curvature at any point of the curve of intersection of two coaxal quadrics, the coordinates being expressed as in Ex. 5, Art. 361.

Ex. 2. The direction-cosines of the binormal (λ, μ, ν) are

$$\lambda = \rho\frac{dyd^2z - dzd^2y}{ds^3}, \text{ \&c.},$$

and of the principal normal (l, m, n) are

$$l = \rho\frac{dsd^2x - dxd^2s}{ds^3}, \text{ \&c.}$$

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since the projections of BC on the axes are $dx + d^2x$, $dy + d^2y$, $dz + d^2z$, it is plain that the projections on the axes of the diagonal BD are d^2x, d^2y, d^2z, whence $BD^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2$. But BD projected on the element of the arc is d^2s, and on a line perpendicular to it is $ds\,d\theta$; whence

$$(d^2s)^2 + (ds\,d\theta)^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2.$$

Ex. 3. The equations of the polar line at x', y', z' are

$$\frac{x-\rho l-x'}{\lambda}=\frac{y-\rho m-y'}{\mu}=\frac{z-\rho n-z'}{\nu}.$$

Express also the coordinates of the tangent plane to the polar developable in terms of a single parameter.

Ex. 4. Find the centre of curvature of the circular helix, and prove that its locus is a circular helix having the same axis and the same angle.]

367. *When a curve is given as the intersection of two surfaces which cut at right angles*, an expression for the radius of curvature can be easily obtained. Let r and r' be the radii of curvature of the normal sections of the two surfaces, the sections being made along the tangent to the curve; and let ϕ be the angle which the osculating plane makes with the first normal plane: then, by Meunier's theorem we have $\rho=r\cos\phi$, and also $\rho=r'\sin\phi$, whence

$$\frac{1}{\rho^2}=\frac{1}{r^2}+\frac{1}{r'^2}.$$

The same equations determine the osculating plane by the formula $\tan\phi=\dfrac{r}{r'}$.

If the angle which the surfaces make with each other be ω, the corresponding formula is

$$\frac{\sin^2\omega}{\rho^2}=\frac{1}{r^2}+\frac{1}{r'^2}-\frac{2\cos\omega}{rr'}.$$

We can hence obtain an expression for the *radius of curvature of a curve given as the intersection of two surfaces.* We may write $L^2+M^2+N^2=R^2$, $L'^2+M'^2+N'^2=R'^2$; and we have

$$\cos\omega=\frac{LL'+MM'+NN'}{RR'},$$

$$\sin^2\omega=\frac{(MN'-M'N)^2+(NL'-N'L)^2+(LM'-L'M)^2}{R^2R'^2}.$$

We must then substitute in the formula of Art. 296,

$$\cos\alpha=\frac{MN'-M'N}{RR'\sin\omega},\ \cos\beta=\frac{NL'-N'L}{RR'\sin\omega},\ \cos\gamma=\frac{LM'-L'M}{RR'\sin\omega}.$$

The denominator of that formula becomes

$$\begin{vmatrix} a, & h, & g, & L, & L' \\ h, & b, & f, & M, & M' \\ g, & f, & c, & N, & N' \\ L, & M, & N, & & \\ L', & M', & N', & & \end{vmatrix}$$

which reduced, as in Art. 362, becomes $\frac{1}{(m-1)^2}S$: giving

$$r = \frac{(m-1)^2 R^3 R'^2 \sin^2\omega}{S}, \quad \text{similarly} \quad r' = \frac{(n-1)^2 R^2 R'^3 \sin^2\omega}{S'}.$$

Whence
$$\frac{1}{\rho^2} = \frac{S^2}{(m-1)^4 R^6 R'^4 \sin^6\omega}$$
$$+ \frac{S'^2}{(n-1)^4 R^4 R'^6 \sin^6\omega} - \frac{2SS' \cos\omega}{(m-1)^2(n-1)^2 R^5 R'^5 \sin^6\omega}.$$

In the notation of Art. 363 this may be written

$$\frac{R^4 R'^4 \sin^6\omega}{\rho^2} = \frac{T^2}{R^2} + \frac{T'^2}{R'^2} - \frac{2TT' \cos\omega}{RR'}.$$

Ex. Find the radius of curvature of a line of curvature of a central quadric, this being the curve of intersection with a confocal quadric.

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1.$$

368. Let us now consider the angle made with each other by two consecutive osculating planes, which we shall call the *angle of torsion*, and denote by $d\eta$. The direction-cosines of the osculating plane being proportional to X, Y, Z, the second formula of Art. 358 gives

$$(X^2 + Y^2 + Z^2)^2 d\eta^2 = (YdZ - ZdY)^2 + (ZdX - XdZ)^2$$
$$+ (XdY - YdX)^2.$$

Now
$$Y = dzd^2x - dxd^2z, \quad Z = dxd^2y - dyd^2x,$$
$$dY = dzd^3x - dxd^3z, \quad dZ = dxd^3y - dyd^3x.$$

Therefore (*Lessons on Higher Algebra*, Art. 31)

$$YdZ - ZdY = Mdx$$

where M is the determinant

$$Xd^3x + Yd^3y + Zd^3z.$$

Hence
$$(X^2 + Y^2 + Z^2)^2 d\eta^2 = M^2 ds^2,$$
$$d\eta = \frac{Mds}{X^2 + Y^2 + Z^2}.$$

This formula may be also proved geometrically. For M denotes six times the volume of the pyramid made by four consecutive points, while $X^2 + Y^2 + Z^2$ denotes four times the square of the area of the triangle formed by three consecutive points. Now if A be the triangular base of a pyramid, A' an adjacent face making an angle η with the base, s the side common to the two faces, and p the perpendicular from the vertex on s, so that $2A' = sp$, then for the volume of the pyramid we have $3V = Ap \sin\eta$ and $6Vs = 2Aps \sin\eta = 4AA' \sin\eta$. Now, in the case considered, the common side is ds, and in the limit $A = A'$; hence $6Vds = 4A^2 d\eta$. Q.E.D.

Following the analogy of the radius of curvature which is $\frac{ds}{d\theta}$, the later French writers denote the quantity * $\frac{ds}{d\eta}$ by the letter τ, and call it the *radius of torsion* its reciprocal being the *torsion;* but the reader will observe that this is not, like the radius of curvature, the radius of a real circle intimately connected with the curve. [The geometrical meaning of the torsion consists in the fact that it measures the divergence of the curve from a plane, being equal to zero for a plane curve.]

[Ex. If the curvature vanishes $X = Y = Z = 0$. Find a determinate formula for the torsion in this case, taking ds^3 into account.]

[368*a*. *The Frenet-Serret formulas.* The differentials with regard to s, of the direction-cosines of the tangent line (α, β, γ), of the principal normal (l, m, n) and of the bi-normal (λ, μ, ν) may be expressed in terms of these cosines and the radii of curvature and torsion by a set of useful formulas. These are :—

$$\frac{d\alpha}{ds} = \frac{l}{\rho}, \quad \frac{d\beta}{ds} = \frac{m}{\rho}, \quad \frac{d\gamma}{ds} = \frac{n}{\rho} \quad \dots \quad (1)$$

$$\frac{d\lambda}{ds} = \frac{l}{\tau}, \quad \frac{d\mu}{ds} = \frac{m}{\tau}, \quad \frac{d\nu}{ds} = \frac{n}{\tau} \quad \dots \quad (2)$$

$$\frac{dl}{ds} = -\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right), \quad \frac{dm}{ds} = -\left(\frac{\beta}{\rho} + \frac{\mu}{\tau}\right), \quad \frac{dn}{ds} = -\left(\frac{\gamma}{\rho} + \frac{\nu}{\tau}\right) \quad \dots \quad (3)$$

The first have been proved in Art. 366, since $\alpha = \frac{dx}{ds}$, &c.

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* The quantity $\frac{d\eta}{ds}$ is also sometimes called the "second curvature" of the curve.

Again, since $\tau = \frac{ds}{d\eta}$ where $d\eta$ is the angle between λ, μ, ν and $\lambda + d\lambda$, $\mu + d\mu$, $\nu + d\nu$, we have

$$\frac{1}{\tau^2} = \left(\frac{d\lambda}{ds}\right)^2 + \left(\frac{d\mu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)^2.$$

To find $\frac{d\lambda}{ds}$ we differentiate the equations

$$\lambda^2 + \mu^2 + \nu^2 = 1,\ a\lambda + \beta\mu + \gamma\nu = 0$$

and we get

$$\lambda\frac{d\lambda}{ds} + \mu\frac{d\mu}{ds} + \gamma\frac{d\nu}{ds} = 0 \text{ and also } a\frac{d\lambda}{ds} + \beta\frac{d\mu}{ds} + \gamma\frac{d\nu}{ds} = 0,$$

by using the first formula along with $l\lambda + m\mu + n\nu = 0$. But we have $l\lambda + m\mu + n\nu = 0$, $la + m\beta + n\gamma = 0$; hence using the above expression for $\frac{1}{\tau}$ we get the second formula.

The third formula is given by differentiating the equations $l = \mu\gamma - \nu\beta$, &c. (in which the sign is fixed by convention), and using the first two formulas.

The above formulas may be expressed in a compact form as follows If p, q, r are the direction-cosines of any fixed line in space, referred to the tangent line, principal normal and bi-normal as moving axes, then

$$\frac{dp}{ds} = \frac{q}{\rho},\ \frac{dq}{ds} = -\left(\frac{p}{\rho} + \frac{r}{\tau}\right),\ \frac{dr}{ds} = \frac{q}{\tau}.$$

Assumptions have been made in the preceding with regard to signs, and these have to be fixed by consistent conventions. The signs of ρ and τ are ambiguous since they involve square roots. We choose any arbitrary direction for the positive direction of the tangent line PT, we fix the positive direction of the principal normal PN by the convention that ρ is always positive, and is measured along PN. The positive direction of the bi-normal PB is now fixed by supposing that the orientation of the axes PT, PN, PB is the same as that of three fixed arbitrary axes of x, y, z; i.e. the former set can be superimposed on the latter by rotation and translation. This implies that we must take the signs $a = m\nu - n\mu$, &c., $l = \mu\gamma - \nu\beta$, &c., $\lambda = \beta n - \gamma m$, &c., for if $a = +1$, and $m = +1$, we must have $\nu = +1$. See Ex. 1, Art. 32.

The sign of τ is now fixed by one of the equations $\frac{d\lambda}{ds} = \frac{l}{\tau}$, &c. If the coordinate axes form a right-handed system (that is, if the rotation through a right angle about Oz which brings Ox into the position originally occupied by Oy is counter-clockwise as seen from the positive side of the plane xOy) the curve is said to be *sinistrorsum* or *dextrorsum* according as the torsion is positive or negative.

Ex. 1. Taking the origin on the curve and the tangent, principal normal, and bi-normal for the axes of x, y, z, the coordinates of points near the origin may be expanded by Maclaurin's theorem in the form

$$x = s - \frac{1}{6\rho^2}s^3 + \ldots$$
$$y = \frac{s^2}{2\rho} - \frac{1}{6\rho^2}\frac{d\rho}{ds}s^3 + \ldots$$
$$z = -\frac{1}{6\rho\tau}s^3 + \ldots$$

The sign of torsion is the same as that of $-\frac{z}{s^3}$ since ρ is positive by convention. Hence if a point moving along a curve in the positive direction is passing at its momentary position from the negative to the positive side of the osculating plane, the torsion is negative (*dextrorsum*) since z is positive, but if the point is passing from the positive to the negative side, the torsion is positive (*sinistrorsum*).

Ex. 2. The torsion as well as the curvature of a circular helix is constant. The radius of torsion is $\pm\frac{a^2+h^2}{h} = \pm\frac{a}{\sin\alpha\cos\alpha}$, where α is the angle the tangent to the helix makes with the axis of the cylinder.

The difference between *sinistrorsum* and *dextrorsum* may be illustrated by comparing a right-handed and left-handed helix on the same cylinder. The first is *dextrorsum*, the second is *sinistrorsum*; the sign of torsion depends on the chosen direction of the bi-normal. If two helices on the same cylinder have the same angle their form and dimensions are the same, but the one cannot be superimposed on the other unless the sign of torsion is the same.

Ex. 3. Prove from the expansion in Ex. 1 that the osculating plane, provided $\frac{1}{\tau}$ is not equal to zero, crosses the curve at the point. Determine the form of the orthogonal projections of the curve on the three coordinate planes in the neighbourhood of the point.

Ex. 4. A *cylindrical helix* is a geodesic on a cylinder and it cuts the generators at a constant angle. Using the Frenet-Serret formulas, and taking the axis of z parallel to the axis of the cylinder prove (a) that the principal normal coincides with the normal to the cylinder, (b) that the binormal makes a constant angle ($\cos^{-1}\nu$) with the axis, and (c) that *the ratio of the curvature to the torsion is constant* and equals $\pm\tan A$ where A is the angle between tangent line and generator.

Ex. 5. Conversely it may be proved that *if the ratio $\frac{\tau}{\rho}$ is constant the curve is a cylindrical helix.**

Let $\tau = k\rho$, then by the formulas $\alpha = k\lambda + a$, $\beta = k\mu + b$, $\gamma = k\nu + c$, where a, b, c are constant. Hence $a^2 + \beta^2 + \gamma^2 = a\alpha + b\beta + c\gamma = 1$, and therefore the tangents to the curve make a constant angle with a fixed direction, which is the axis of the helix.

Ex. 6. Puiseux (Liouville's *Monge*, p. 554) has also proved that *if the curvature and torsion be both constant the curve is a circular helix.* More

* [Noticed first by Bertrand, *Journal de Math. pures et appliquées*, 8 (1843).]

generally the curvature of the orthogonal projection of a curve on the plane of z at a projected point may be found For if $x = \phi(s)$, $y = \psi(s)$, $z = \chi(s)$ represent the original curve, its projection is $z = 0$, $x = \phi(s)$, $y = \psi(s)$, where s is the arc of the *original* curve Hence the curvature of the projection at $x, y, 0$ is $\frac{dx\,d^2y - dy\,d^2x}{(dx^2 + dy^2)^{\frac{3}{2}}} = \frac{m\alpha - l\beta}{\rho(1-\gamma^2)^{\frac{3}{2}}} = \frac{\nu}{\rho(1-\gamma^2)^{\frac{3}{2}}}$ If ρ and τ are both constant, and if the axis of z is parallel to the fixed direction (Ex 5), this is constant and the projection is thus a circle

[368*b* The *intrinsic equations of a curve* are relations of the form $\rho = \phi(s)$, $\tau = \psi(s)$ They have the advantage of being independent of any particular axis But it is necessary to prove that a real curve is determined, irrespective of position, by two such equations Suppose the curves 1 and 2 have the same intrinsic equations Place them so that the points for which $s = 0$ coincide, and the tangents, normals, and binormals at this point also coincide By using the Frenet-Serret formulas we easily prove $\frac{d}{ds}(\alpha_1\alpha_2 + l_1 l_2 + \lambda_1\lambda_2) = 0$ where x_1, y_1, z_1 and x_2, y_2, z_2 are points on the two curves corresponding to the same value of s Hence $\alpha_1\alpha_2 + l_1 l_2 + \lambda_1\lambda_2$ is constant But when $s = 0$, $\alpha_1 = \alpha_2$, $l_1 = l_2$, $\lambda_1 = \lambda_2$, therefore the constant equals unity Also $\alpha_1^2 + l_1^2 + \lambda_1^2 = 1$, and $\alpha_2^2 + l_2^2 + \lambda_2^2 = 1$, therefore

$$(\alpha_1 - \alpha_2)^2 + (l_1 - l_2)^2 + (\lambda_1 - \lambda_2)^2 = 0$$

Hence for real curves $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$ Consequently $x_1 - x_2 = \text{constant} = 0$ and similarly $y_1 = y_2$, $z_1 = z_2$, i e the two curves coincide *

A more general form of the preceding theorem is the following *If the points of two curves are in one to one correspondence so that*

$$\tau_2 = p\tau_1,\ \rho_2 = p\rho_1,\ ds_2 = p\,ds_1,$$

where p varies from point to point, the two curves have the same spherical indicatrix (irrespective of position)

We may suppose that the two curves are expressed by a parameter t, which has the same value at corresponding points, then p is a function of t Using

$$\frac{d}{dt} = \frac{ds_1}{dt}\frac{d}{ds_1} = \frac{ds_2}{dt}\frac{d}{ds_2}$$

* It may be proved that the expression of x, y, z in terms of the arc, for a curve given by its intrinsic equations, depends on the solution of a Riccati equation. See Darboux, *Surfaces*, vol I.

and remembering that

$$\frac{1}{\rho_1}\frac{ds_1}{dt} = \frac{1}{\rho_2}\frac{ds_2}{dt}, \quad \frac{1}{\tau_1}\frac{ds_1}{dt} = \frac{1}{\tau_2}\frac{ds_2}{dt},$$

we find as before

$$\frac{d}{dt}(\alpha_1\alpha_2 + l_1l_2 + \lambda_1\lambda_2) = 0.$$

If we place the curves so that the point for which $s_1 = 0$ corresponds to the point $s_2 = 0$, and so that the tangent lines, normals, and bi-normals are respectively coincident, we get as before $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$; also $l_1 = l_2$ &c.; $\lambda_1 = \lambda_2$, &c., proving that the three sets of lines are parallel each to each at corresponding points. The curves have the same spherical indicatrix, for $\alpha_1 = \alpha_2$ therefore $dx_2 = p dx_1$, &c. (See Ex., Art. 358.)

Ex. 1. Prove the converse of the preceding—if two curves have the same spherical indicatrix $\tau_1 : \tau_2 = \rho_1 : \rho_2 = ds_1 : ds_2$.

When two curves have the same spherical indicatrix, the tangents, normals, and bi-normals at corresponding points are parallel. The relation between two such curves is described as a "transformation of Combescure".

Ex. 2. If a curve be traced on a right cone (of semi-vertical angle α) so that the tangent line makes a constant angle β with the generator at the point, its coordinates may be expressed as

$$x = e^{\theta \sin\alpha \cot\beta} \cos\theta \sin\alpha \cos\beta,$$
$$y = e^{\theta \sin\alpha \cot\beta} \sin\theta \sin\alpha \cos\beta,$$
$$z = e^{\theta \sin\alpha \cot\beta} \cos\alpha \cos\beta.$$

Prove that the intrinsic equations of the curve are of the form $\rho = as$, $\tau = bs$, where a and b are constants.

Conversely if the intrinsic equations are of this form, the curve is of the type mentioned.

368c. *Bertrand Curves.* Bertrand has considered the interesting problem, to determine the *condition that the principal normals to a curve C_1 may also be principal normals to another curve C_2.*

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be coordinates of corresponding points P_1, P_2. Let $P_1P_2 = r$; l, m, n being the direction-cosines of the common normal, we have

$$x_2 - x_1 = rl, \text{ \&c.}$$

Differentiating and using $ldx_1 + mdy_1 + ndz_1 = ldx_2 + mdy_2 + ndz_2 = 0$ we find $dr = 0$. Hence *the distance between corresponding points is of constant length.*

By differentiating with regard to s_1 the three equations of the type

$$x_2 = x_1 + rl,$$

we get three equations of the type

$$\frac{dx_2}{ds_1} = \frac{dx_1}{ds_1} + r\frac{dl}{ds_1} = \alpha_1 - r\left(\frac{\alpha_1}{\rho_1} + \frac{\lambda_1}{\tau_1}\right).$$

Squaring and adding and taking the positive square root

$$\frac{ds_2}{ds_1} = \sqrt{\left(1 - \frac{r}{\rho_1}\right)^2 + \frac{r^2}{\tau_1^2}}.$$

Also $\dfrac{dx_2}{ds_2} = \left(\dfrac{dx_1}{ds_1} + r\dfrac{dl}{ds_1}\right)\dfrac{ds_1}{ds_2}$, with three similar equations Therefore using the Frenet-Serret formulas

$$\alpha_2 = \alpha_1 \cos\phi + \lambda_1 \sin\phi$$
$$\beta_2 = \beta_1 \cos\phi + \mu_1 \sin\phi$$
$$\gamma_2 = \gamma_1 \cos\phi + \nu_1 \sin\phi$$

where $\cos\phi = \dfrac{1 - \dfrac{r}{\rho_1}}{\sqrt{\left(1 - \dfrac{r}{\rho_1}\right)^2 + \dfrac{r^2}{\tau_1^2}}}$, $\sin\phi = \dfrac{-\dfrac{r}{\tau_1}}{\sqrt{\left(1 - \dfrac{r}{\rho_1}\right)^2 + \dfrac{r^2}{\tau_1^2}}}$.

Since

$$\cos\phi = \alpha_1\alpha_2 + \beta_1\beta + \gamma_1\gamma_2,$$

ϕ is the angle between corresponding tangent lines

Differentiating with regard to s_1 the equation $\cos\phi - \alpha_1\alpha + \beta_1\beta + \gamma_1\gamma_2$ the result is

$$\Sigma\alpha\frac{d\alpha_1}{ds_1} + \left(\Sigma\alpha_1\frac{d\alpha_2}{ds_2}\right)\frac{ds_2}{ds_1}$$

which by the Frenet Serret formulas is equal to zero Hence *the angle ϕ between corresponding tangent lines is constant*

$$\text{Again } \frac{\cos\phi}{\tau_1} - \frac{\sin\phi}{\rho_1} + \frac{\sin\phi}{r} = 0, \text{ or } \frac{A}{\rho_1} + \frac{B}{\tau_1} = C,$$

where A, B, C are constants

Hence *a linear relation connects the curvature and torsion* of the curve C_1 A corresponding linear relation of course exists for the second curve, and this may be found by changing the signs of ϕ and r when we get

$$\frac{\cos\phi}{\tau_2} + \frac{\sin\phi}{\rho_2} + \frac{\sin\phi}{r} = 0, \text{ or } \frac{A}{\rho_2} - \frac{B}{\tau_2} = -C$$

Again we found $\dfrac{ds_2}{ds_1} = -\dfrac{r}{\tau_1 \sin\phi}$ and the corresponding relation for the second curve,—changing the signs of r and ϕ—is $\dfrac{ds_1}{ds_2} = -\dfrac{r}{\tau_2 \sin\phi}$ Hence $\tau_1\tau_2 = \dfrac{r^2}{\sin^2\phi}$, therefore *the torsions at corresponding points have the same sign*

It may be proved that the existence of a linear relation between the curvature and torsion of a curve C_1 is *a sufficient condition* that it should be a Bertrand curve, the value and sign of r being determined by $r = \dfrac{A}{C}$, and ϕ from the equation $\tan\phi = -\dfrac{A}{B}$

Ex Discuss the cases for which

$$C = 0,\ A = 0\quad A = C = 0,\ \rho = \text{const},\ \tau = \text{const}$$

Given any curve we can find any number of Bertrand curves by transformations of Combescure * (Art 368 *b*, Ex 1) For if x_1, y_1, z_1, s_1 are the ele-

* Bianchi, *Lezioni di Geometria Differenziale* (1902), § 25.

ments of a curve C_1 any curve C_2 having the same spherical indicatrix is given by equations of the form (Art. 358)

$$x_2 = \int \alpha_1 f(s_1)ds_1,\ y_2 = \int \beta_1 f(s_1)ds_1,\ z_2 = \int \gamma_1 f(s_1)ds_1$$

where $f(s_1)$ is an arbitrary function. Also (Art. 368 *b*, Ex. 1)

$$ds_2 = f(s_1)ds_1,\ \frac{ds_2}{\rho_2} = \frac{ds_1}{\rho_1},\ \frac{ds_2}{\tau_2} = \frac{ds_1}{\tau_1}.$$

Suppose $\dfrac{A}{\rho_2} + \dfrac{B}{\tau_2} = C$, then the function $f(s_1)$ is determined by $f(s_1) = \dfrac{1}{C}\left(\dfrac{A}{\rho_1} + \dfrac{B}{\tau_1}\right)$, ρ_1 and τ_1 being given by the intrinsic equations in terms of s_1.]

369. *Ratio of curvature to torsion.* As we have considered an osculating circle determined by three consecutive points of the system, we may consider an *osculating right cone* determined by three consecutive planes of the system, and we proceed to determine its vertical angle. Imagine that a sphere is described having as centre the point of the system in which the three planes intersect; let the lines of the system passing through that point meet the sphere in A and B; and let the corresponding planes meet the same sphere in AT, BT; then, if we describe a small circle of the same sphere touching AT, BT, and escribed to AB, the cone whose vertex is the centre, and which stands on that small circle, will evidently osculate the given curve. The problem then is, being given $d\eta$ the angle between two consecutive tangents to a small circle of a sphere, and $d\theta$ the corresponding arc of the circle, to find H its radius.

Let ϕ be the external angle between two tangents to a circle, s the length of the two tangents, then H the radius of the circle is given by the formula $\tan \tfrac{1}{2}\phi \tan H = \sin \tfrac{1}{2}s$. Now, taking C the centre of the small circle and t the foot of the perpendicular from it on AB, we have $\tan \tfrac{1}{2}\phi \tan H = \sin At$, and $\tan \tfrac{1}{2}\phi' \tan H = \sin Bt$, where in the limit ϕ' differs by an infinitely small quantity from ϕ.

Now, since also in the limit AB measures the angle between consecutive lines of the system and ϕ measures that between consecutive planes of the system, we have then

$$\tan H = \frac{d\theta}{d\eta} = \frac{\tau}{\rho}.$$

[The value of H and the direction-cosines of the axis of the cone (p, q, r) may be found analytically by using the Frenet-Serret formulas. The axis of the cone makes equal angles with "three consecutive" bi-normals, therefore

$$p\lambda + q\mu + r\nu = \sin H.$$

$$p\frac{d\lambda}{ds} + q\frac{d\mu}{ds} + r\frac{d\nu}{ds} = 0$$

$$p\frac{d^2\lambda}{ds^2} + q\frac{d^2\mu}{ds^2} + r\frac{d^2\nu}{ds^2} = 0.$$

Using the formulas referred to, with $p^2 + q^2 + r^2 = 1$, we find

$$p = \frac{\lambda\tau - a\rho}{\sqrt{\tau^2 + \rho^2}}, \text{ \&c.,}$$

$$\text{and } \sin H = \frac{\tau}{\sqrt{\tau^2 + \rho^2}}.]$$

370. [There are three developables especially associated with the curve, namely the envelopes of the planes containing any two out of the three lines, the tangent line, the binormal and the principal normal. One of these is the developable generated by tangent lines, another is the polar developable; and the third we shall now consider. That these surfaces are developables is clear from the fact that the coordinates of their planes involve only one parameter, viz. that corresponding to the points on the curve.] Imagine that through every line of the system there is drawn a plane perpendicular to the corresponding osculating plane, this is called a *rectifying plane*, and the assemblage of these planes generates a developable which is called the *rectifying* developable. The reason of the name is, that *the given curve is obviously a geodesic on this developable,* since its osculating plane is, by construction, everywhere normal to the surface. *If, therefore, the developable be developed into a plane, the given curve will become a right line.*

The intersection of two consecutive planes of the rectifying developable is the *rectifying line.* Now, since the plane passing through the edge of a right cone perpendicular to its tangent plane passes through its axis, it follows that the rectifying plane passes through the axis of the osculating cone considered in the last article; and, therefore, that *the rectifying line is the axis of that osculating cone.* The rectifying line may be therefore constructed by drawing in the

rectifying plane a line making with the tangent line an angle H, where H has the value determined in the last article.

The rectifying surface is the surface of centres of the original developable formed by the lines of the system. In fact it was proved (Art. 306) that the normal planes to a surface along the two principal tangents touch the surface of centres; but for a developable (which has the same tangent plane at every point on a generator, and one centre of curvature therefore at infinity) the generating line itself is in every point of it one of the principal tangents; the rectifying plane, therefore, touches the surface of centres which is the envelope of all these rectifying planes. The centre of curvature at any point on a developable of the other principal section, namely, that perpendicular to the generating line, is the point where its plane meets the corresponding rectifying line; for evidently the traces on this plane of two consecutive rectifying planes are two consecutive normals to the section. *Hence if l be the distance of any point on the developable from the cuspidal edge measured along the generator, the radius of curvature of the transverse section is l* tan H. When l vanishes, this radius of curvature vanishes, as it ought, the point being a cusp.

In the case of the helix the rectifying surface is obviously the cylinder on which the curve is traced.

[Ex. The coordinates of the edge of regression of the rectifying developable, corresponding to a point x, y, z on the curve are $x + \dfrac{\alpha\rho - \lambda\tau}{\frac{d\tau}{ds}\rho - \frac{d\rho}{ds}\tau}\tau$, &c. Hence this curve is expressed by means of the parameter of the original curve.]

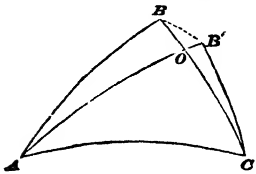
371. *To find the angle between two successive radii of curvature.**

Let AB, BC be traces on any sphere with radius unity, of planes parallel to the osculating and normal planes, then

* The reader will find simple geometrical investigations of this and other formulæ connected with curves of double curvature in a paper by Routh, *Quarterly Journal of Mathematics*, Vol. VII. p. 37.

the central radius to B is the direction of the radius o curvature. If AB', $B'C$ be consecutive positions of the osculating and normal planes, B' is in the direction of the consecutive radius of curvature, and BB' measures the angle between them. Now the triangle BOB' being a very small right-angled triangle, we have $BB'^2 = BO^2 + OB'^2$.

But since the angle ABC is right, BO measures BAB', which is $d\eta$, the angle between two consecutive osculating planes, and OB' measures OCB', which is $d\theta$, the angle between two consecutive normal planes. *The required angle is therefore given by the formula* $BB'^2 = d\eta^2 + d\theta^2$, where $d\eta$ and $d\theta$ have the values already found. This also follows directly from the Frenet-Serret formulas; for if $d\chi$ is the angle, between l, m, n and $l + dl$, $m + dm$, $n + dn$, $d\chi^2 = dl^2 + dm^2 + dn^2$, therefore $\left(\frac{d\chi}{ds}\right)^2 = \frac{1}{\rho^2} + \frac{1}{\tau^2}$. The series of radii of curvature at all the points of a curve generate a surface on the properties of which we have not space to dwell. It is evidently a skew surface (see note, Art. 112) since two consecutive radii do not in general intersect (see Art. 374, *infra*).



Ex. 1. To find the equation of the surface of the radii of curvature in the case of the circular helix.

The radius of curvature being the intersection of the osculating and normal planes has for its equations (Art. 361) $x'y = y'x$, $z = z'$, from which we are to eliminate $x'y'z'$ by the help of the equations of the curve. And writing the equations of the helix $x = a \cos nz$, $y = a \sin nz$, the required surface is $y \cos nz = x \sin nz$.

[In general the equation of this surface for any curve is found by eliminating the parameter from the equations

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n}.\ ]$$

Ex. 2. To find the equation of the developable generated by the tangents of a circular helix. The equations of the tangent being

$$x - a \cos nz' = - na \sin nz'(z - z'),\ y - a \sin nz' = na \cos nz'(z - z'),$$

the result of eliminating z' is found to be

$$x \cos \left\{nz \pm \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} + y \sin \left\{nz \pm \frac{(x^2 + y^2 - a^2)}{a}\right\} = a.$$

Since this equation becomes impossible when $x^2 + y^2 < a^2$, it is plain that no part of the surface lies within the cylinder on which the helix is traced.

372. We shall now speak of the *polar developable* generated by the normal planes to the given curve. Fourier has remarked, that the "angle of torsion" of the one system is equal to the "angle of contact" of the other, as is sufficiently obvious since the planes of this new system are perpendicular to the lines of the original system, and *vice versâ*. The reader will bear in mind, however, that it does not follow from this that the $\frac{d\theta}{ds}$ of one system is equal to the $\frac{d\eta}{ds}$ of the other, because the ds is not the same for both.

Since the intersection of the normal planes at two consecutive points K, K' of the curve is the axis of a circle of which K and K' are points (Art. 364), it follows that if any point D on that line be joined to K and K', the joining lines are equal and make equal angles with that axis.

It is plain that three consecutive normal planes intersect in the centre of the osculating sphere; hence *the cuspidal edge of the polar developable is the locus of centres of spherical curvature.*

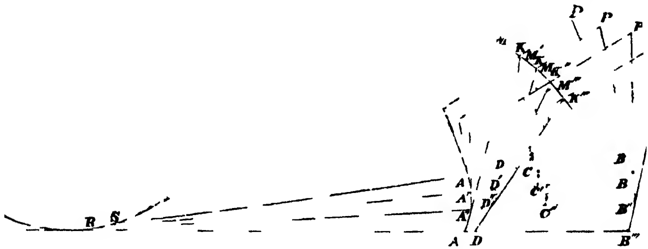
In the case of a plane curve this polar developable reduces to a cylinder standing on the evolute of the curve.

373. *Every curve has an infinity of evolutes lying on the polar developable;** that is to say, the given curve may be generated in an infinity of ways by the unrolling of a string wound round a curve traced on that developable. Let MM', $M'M''$, &c. denote the successive elements of the curve, K, K', &c. the middle points of these elements, then the planes drawn through the points K perpendicular to the elements are the normal planes. The lines AB, $A'B'$, &c. being the lines in which each normal plane is intersected by the consecutive, these lines are the generators of the polar developable, and hence tangents to the cuspidal edge RS of that

* See Monge, p. 396.

surface Draw now at pleasure * any line KD in the first normal plane, meeting the first generator in D; join DK' which being in the second normal plane will meet the second generator $A'B'$, say in D' In like manner, let $K''D'$ meet $A''B''$ in D'' We get thus a curve $DD'D''$ traced on the polar developable which is an evolute of the given curve. For the lines DK, $D'K'$, &c the tangents to the curve $DD'D''$, are normals to the curve $KK'K''$, and the lengths $DK = DK'$, $D'K' = D'K''$, &c (see Art 372) If therefore DK be a part of a thread wound round $DD'D''$, it is plain that as the thread is unwound the point K will move along the given curve

Since the first line DK was arbitrary, the curve has an infinity of evolutes. A plane curve has thus an infinity of



evolutes lying on the cylinder whose base is the evolute in the plane of the curve For example, in the special case where this evolute reduces to a point that is, when the curve is a circle, the circle can be described by moving round a thread of constant length fastened to any point on the axis passing through the centre of the circle

In the general case, *all the evolute curves $DD'D''$ &c are geodesics on the polar developable*

For we have seen (Art 308) that a curve is a geodesic when two successive tangents to it make equal angles with the intersection of the corresponding tangent planes of the surface, and it has just been proved (Art 372), that

* This figure is taken from Leroy's *Geometry of Three Dimensions*

DK, DK', which are two successive tangents to the evolute, make equal angles with AB which is the intersection of two consecutive tangent planes of the developable. An evolute may then be found by drawing a thread as tangent from K to the polar developable, and winding the continuation of that tangent freely round the developable.

374. The locus of centres of curvature is a curve on the polar developable, but generally is *not* one of the system of evolutes. Let the first osculating plane $MM'M''$ meet the first two normal planes in KC, $K'C$, then C is the first centre of curvature; and, in like manner, the second centre is C', the point of intersection of $K'C'$, $K''C'$, the lines in which the second osculating plane $M'M''M'''$ is met by the second and third normal planes. Now the radii $K'C$, $K'C'$ are distinct, since they are the intersections of the same normal plane by two different osculating planes, $K'C'$ will therefore meet the line AB in a point I which is distinct from C. Consequently, the two radii of curvature KC, $K'C'$ situated in the planes P, P' have no common point in AB the intersection of these planes; two consecutive radii therefore do not intersect, unless in the case where two consecutive osculating planes coincide.

The centres of curvature then not being given by the successive intersections of consecutive radii, these radii are not tangents to the locus of centres. Any radius therefore KC would not be the continuation of a thread wound round $CC'C''$, and the unwinding of such a thread would not give the curve $KK'K''$, except in the case where the latter is a plane curve.*

* The characteristics of the polar developable may be investigated by arguments similar to those used, *Higher Plane Curves*, Arts. 111, &c. They are $n' = m + r$, $a' = 0$, $r' = 3m + n$, $m' = 5m + a$, where m, n, &c., having the same meaning as in Art. 325, are the characteristics of the given curve, and m', n', &c. the corresponding characteristics of the polar developable. When, as is here supposed, there is nothing special in the character of the points at infinity of the given curve, the normal planes corresponding to these points are altogether at infinity; and the corresponding generators of the polar

375. *To find the radius of the sphere through four consecutive points.* Let R be the radius of any sphere, ρ the radius of a section by a plane making an angle η with the normal plane at any point; then, by Meunier's theorem $R \cos \eta = \rho$; and for a consecutive plane making an angle $\eta + \delta\eta$, we have $\delta\rho = -R \sin \eta \delta\eta$. Hence $R^2 = \rho^2 + \left(\frac{d\rho}{d\eta}\right)^2$.

We have then only to give in this expression to ρ and $d\eta$ the values already found.

$\frac{d\rho}{d\eta}$ is obviously the length of the perpendicular distance from the centre of the sphere to the plane of the circle of curvature.

376. *To find the coordinates of the centre of the osculating sphere.*

Let the equation of any normal plane be

$$(\alpha - x)\, dx + (\beta - y)\, dy + (\gamma - z)\, dz = 0,$$

where xyz is the point on the curve, and $\alpha\beta\gamma$ any point on the plane; then the equation of a consecutive normal plane combined with the preceding gives

$$(\alpha - x)\, d^2x + (\beta - y)\, d^2y + (\gamma - z)\, d^2z = ds^2.$$

And the equation of the third plane gives

$$(\alpha - x)\, d^3x + (\beta - y)\, d^3y + (\gamma - z)\, d^3z = 3ds d^2s.$$

Let us denote, as before, $dyd^2z - dzd^2y$, &c. by, X, Y, Z; $dyd^3z - dzd^3y$, &c. by X', Y', Z', and the determinant $Xd^3x + Yd^3y + Zd^3z$ by M. Then, solving the preceding equations, we have

$$M(\alpha - x) = -X'ds^2 + 3Xdsd^2s,\quad M(\beta - y) = -Y'ds^2 + 3Ydsd^2s,$$
$$M(\gamma - z) = -Z'ds^2 + 3Zdsd^2s.$$

By squaring and adding these equations, we obtain another expression for R^2, which is what the value in the last article becomes when for ρ and $\frac{d\rho}{d\eta}$ we substitute their values.

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developable are common to three consecutive planes. The plane at infinity meets the polar developable in m lines, each reckoned three times, and a curve of the n^{th} order.

We add a few other expressions, the greater part of which admit of simple geometrical proofs, the details of which want of space obliges us to omit.

Ex. 1. If σ be the arc of the curve which is the locus of centres of absolute curvature,

$$d\sigma^2 = d\rho^2 + \rho^2 d\eta^2\,;\ \text{or}\ d\sigma = R d\eta.$$

Ex. 2. If Σ be the length of the arc of the locus of centres of spherical curvature, $d\Sigma = \frac{RdR}{\delta}$; where $\delta = \frac{d\rho}{d\eta}$ is the distance between the centres of the osculating circle and osculating sphere. From this expression we immediately get values for the radii of curvature and of torsion of this locus, remembering that the angle of torsion is the angle of contact of the original, and *vice versâ*.

Ex. 3. The angle between two consecutive rectifying lines is dH.

Ex. 4. The angle ψ between two consecutive R's is given by the formula

$$R^2\psi^2 = ds^2 + d\Sigma^2 - dR^2.*$$

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* The reader will find further details on the subjects treated of in this section in a Memoir by Saint-Venant, *Journal de l'École Polytechnique*, Cahier xxx., who has also collected into a table about a hundred formulæ for the transformation and reduction of calculations relative to the theory of non-plane curves; and in a paper by Frenet, *Liouville*, Vol. XVII. p. 437. I abridge the following historical sketch from Saint-Venant's Memoir: "Curve lines not contained in the same plane have been successively studied by Clairaut (*Recherches sur les courbes à double courbure*, 1731), who has brought into use the title by which they have been commonly known (previously, however, employed by Pitot) and who has given expressions for the projections of these curves, for their tangents, normals, arc, &c.; by Monge (*Mémoire sur les développées, &c.* presented in 1771, and inserted in Vol. X., 1785, of the *Savants étrangers*, as well as in his *Application de l'Analyse à la Géométrie*) who gave expressions for the normal plane, centre and radius of curvature, evolutes, polar lines and polar developable, centre of osculating sphere, for the criterion for 'points of simple inflexion' where four consecutive points are in a plane, and for 'points of double inflexion' where three consecutive points are in a right line; by Tinseau (*Solution de quelques problèmes, &c.* presented in 1774, *Savants étrangers*, Vol. IX., 1780) who was the first to consider the osculating plane and the developable generated by the tangents; by Lacroix (*Calcul Différentiel*) who was the first to render the formulæ symmetrical by introducing the differentials of the three coordinates; and by Lancret (*Mémoire sur les courbes à double courbure*, read 1802, and inserted Vol. I., 1805, of *Savants étrangers* de l'Institut) who calculated the angle of torsion, and introduced the consideration of the rectifying lines and rectifying surface." The reader will find some interesting and novel researches respecting curves of double curvature in Hamilton's *Elements of Quaternions*; as, for instance, the theory of the osculating twisted cubic which passes through six consecutive points of the curve. [See also p. 367, note.]

Section IV. Curves traced on Surfaces.

377. The coordinates x, y, z of a point on a surface may be expressed as functions of two parameters p, q; and conversely if the coordinates x, y, z are thus expressed as functions of two parameters, these expressions determine the surface, for by the elimination of the parameter we obtain between the coordinates x, y, z the equation $U=0$ of the surface; and when a definite value is assigned to either p or q, the point xyz is restricted to a definite curve on the surface. This mode of representation of a surface is peculiarly appropriate for the discussion of the theory of curvature, and it has been used for that purpose by Gauss.* We proceed to give an account of his investigations, but before doing so must explain his notation and establish the connexion of this method with that by which curvature was treated in Chapter XI. We have x, y, z given functions of p, q; and the partial differential coefficients of x, y, z in regard to these variables are expressed as follows:

$$dx = adp + a'dq,\ dy = bdp + b'dq,\ dz = cdp + c'dq,$$
$$d^2x = \alpha dp^2 + 2\alpha' dpdq + \alpha'' dq^2,$$
$$d^2y = \beta dp^2 + 2\beta' dpdq + \beta'' dq^2,$$
$$d^2z = \gamma dp^2 + 2\gamma' dpdq + \gamma'' dq^2.$$

Gauss also writes

$$bc' - cb' = A,\ ca' - ac' = B,\ ab' - ba' = C,$$
$$a^2 + b^2 + c^2 = E,\ aa' + bb' + cc' = F,\ a'^2 + b'^2 + c'^2 = G,$$

which obviously lead to the relation $A^2 + B^2 + C^2 = EG - F^2$; and to these notations it is convenient to join $V^2 = EG - F^2$, $A\alpha + B\beta + C\gamma = E'$, $A\alpha' + B\beta' + C\gamma' = F'$, $A\alpha'' + B\beta'' + C\gamma'' = G'$, E', F', G' denoting respectively the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

* See his Memoir "Disquisitiones circa superficies curvas," *Comm. Gott. recent.*, t. VI. (1827), reprinted in the appendix to Liouville's Edition of Monge, and in his Works, IV. p. 219.

The identity $Adx + Bdy + Cdz = 0$,
replaces the differential equation of the surface, or what is the same thing, if $U = f(x, y, z) = 0$ is the equation of the surface, then A, B, C are proportional to $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$.

Again, since the coordinates are rectangular, if ds be an *element of length on the surface*, that is, if it be the distance between the points (p, q) and $(p + dp, q + dq)$, then

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

378. The differential equation (Art. 303) of the lines of curvature may be written

$$\begin{vmatrix} dx, & dy, & dz \\ A, & B, & C \\ dA, & dB, & dC \end{vmatrix} = 0.$$

Repeating the investigation which led to this equation, we have for the coordinates of an indeterminate point on the normal

$$\xi = x + A\lambda,\ \eta = y + B\lambda,\ \zeta = z + C\lambda,$$

and if this meets the consecutive normal, then taking ξ, η, ζ to be the coordinates of the point of intersection, we have

$$0 = dx + Ad\lambda + \lambda dA,\ 0 = dy + Bd\lambda + \lambda dB,\ 0 = dz + Cd\lambda + \lambda dC,$$

which equations, by eliminating λ and $d\lambda$, give the equation in question.

Now this equation may be written (*Higher Algebra*, Art. 24)

$$\begin{vmatrix} adx + bdy + cdz, & a'dx + b'dy + c'dz \\ adA + bdB + cdC, & a'dA + b'dB + c'dC \end{vmatrix} = 0,$$

since it is what is denoted by

$$\begin{Vmatrix} a, & b, & c \\ a', & b', & c' \end{Vmatrix} \cdot \begin{Vmatrix} dx, & dy, & dz \\ dA, & dB, & dC \end{Vmatrix} = 0.$$

Calculating the quantity $adx + bdy + cdz$, by substituting for dx, $adp + a'dq$, &c., it is found to be $Edp + Fdq$. Similarly

$$a'dx + b'dy + c'dz = Fdp + Gdq.$$

Again, differentiating the identities

$$a\,A + b\,B + c\,C = 0,$$
$$a'A + b'B + c'C = 0,$$

we find $a\,dA + b\,dB + c\,dC = -(Ada + Bdb + Cdc)$,

$a'dA + b'dB + c'dC = -(Ada' + Bdb' + Cdc')$,

which, substituting $adp + a'dq$ for da, &c., become respectively $-(E'dp + F'dq)$ and $-(F'dp + G'dq)$. Whence, finally, *the equation of the lines of curvature is*

$$\begin{vmatrix} Edp + Fdq, & Fdp + Gdq \\ E'dp + F'dq, & F'dp + G'dq \end{vmatrix} = 0,$$

or, as this may also be written,

$$\begin{vmatrix} dq^2, & -dpdq, & dp^2 \\ E, & F, & G \\ E', & F', & G' \end{vmatrix} = 0.$$

379. The equations $0 = dx + Ad\lambda + \lambda dA$, &c., of the last article may be written, putting $dA = A_1dp + A_2dq$, &c.,

$$0 = (a + \lambda A_1)dp + (a' + \lambda A_2)dq + Ad\lambda,$$
$$0 = (b + \lambda B_1)dp + (b' + \lambda B_2)dq + Bd\lambda,$$
$$0 = (c + \lambda C_1)dp + (c' + \lambda C_2)dq + Cd\lambda,$$

which equations, by the elimination of dp, dq, $d\lambda$, give for the determination of λ a quadratic equation corresponding to that of Art. 295. Taking ρ for the radius of curvature, we have $\rho^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = V^2\lambda^2$, or say $\lambda = \rho : V$; and writing down the equation in question with this value substituted for λ, the equation is

$$\begin{vmatrix} aV + A_1\rho, & bV + B_1\rho, & cV + C_1\rho \\ a'V + A_2\rho, & b'V + B_2\rho, & c'V + C_2\rho \\ A, & B, & C \end{vmatrix} = 0,$$

a quadratic equation for determining the radius of curvature. This equation may be treated as before. It becomes

$$\begin{vmatrix} EV + \rho(A_1a + B_1b + C_1c), & FV + \rho(A_1a' + B_1b' + C_1c') \\ FV + \rho(A_2a + B_2b + C_2c), & GV + \rho(A_2a' + B_2b' + C_2c') \end{vmatrix} = 0.$$

In which, by the last article, the coefficients of ρ are $-E'$, $-F'$, $-G'$, whence *the equation for the radii of curvature is*

$$\begin{vmatrix} E'\rho - EV, & F'\rho - FV \\ F'\rho - FV, & G'\rho - GV \end{vmatrix} = 0.$$

380. By what precedes we have a quadratic equation for the directions of the lines of curvature, and a quadratic equation

for the values of ρ; but it is obvious that, selecting at pleasure either of the two lines of curvature, the corresponding value of ρ should be linearly determined. The required formula is at once obtained from the equations $0 = dx + Ad\lambda + \lambda dA$, &c., of Art. 378, by multiplying them by dx, dy, dz respectively and adding; then substituting for λ its foregoing value $\rho : V$, we have

$$V\,(dx^2 + dy^2 + dz^2) + \rho\,(dxdA + dydB + dzdC) = 0$$

where, by what precedes, $dx^2 + dy^2 + dz^2 = Edp^2 + 2Fdpdq + Gdq^2$. But, by the equation of the surface $Adx + Bdy + Cdz = 0$, we have

$$dAdx + dBdy + dCdz = -\,(Ad^2x + Bd^2y + Cd^2z),$$

which, substituting from Art. 377

$$= -\,(E'dp^2 + 2F'dpdq + G'dq^2),$$

whence the equation is

$$\rho\,(E'dp^2 + 2F'dpdq + G'dq^2) - V\,(Edp^2 + 2Fdpdq + Gdq^2) = 0.$$

In this, considering $dp \div dq$ as having at pleasure one or other of the values given by the differential equation of the lines of curvature, the equation gives linearly the corresponding value of the radius of curvature.

But writing the equation in the form

$$(\rho E' - VE)\,dp^2 + 2\,(\rho F' - VF)\,dpdq + (\rho G' - VG)\,dq^2 = 0,$$

and attending to the equation for the determination of ρ, it appears that the equation may be expressed in either of the forms

$$(\rho E' - VE)\,dp + (\rho F' - VF)\,dq = 0,$$
$$(\rho F' - VF)\,dp + (\rho G' - VG)\,dq = 0;$$

or, which is the same thing, the equations of Arts. 378 and 379 may be expressed in the more complete forms

$$\left\| \begin{matrix} \rho\,, & E\,dp + F\,dq, & F\,dp + G\,dq \\ V, & E'dp + F'dq, & F'dp + G'dq \end{matrix} \right\| = 0,$$

$$\left\| \begin{matrix} dq, & \rho E' - VE, & \rho F' - VF \\ -\,dp, & \rho F' - VF, & \rho G' - VG \end{matrix} \right\| = 0.$$

The first of these gives the quadratic equation for the curves of curvature, and (linearly) the value of ρ for each curve; the second gives the quadratic equation for the radius of curvature,

and (linearly) the direction of the curvature for each value of the radius. It also appears that the quadratic equations for ρ and for $dp \div dq$ are linear transformations the one of the other.

381. Returning to the equation

$$\rho\ (E'dp^2 + 2F'dpdq + G'dq^2) = V\ (Edp^2 + 2Fdpdq + Gdq^2)$$

of the preceding article, it is to be observed that (the ratio $dp \div dq$ being arbitrary) this is the equation which *determines the radius of curvature of the normal section through the consecutive point* $(p + dp,\ q + dq)$. The centre of curvature of this section is, in fact, given as the intersection of the normal at (p, q) by the plane drawn through the middle point of the line joining the two points (p, q), $(p + dp,\ q + dq)$ at right angles to this line. Taking ξ, η, ζ for current coordinates, the equations of the normal are, as before,

$$\xi = x + \lambda A,\ \eta = y + \lambda B,\ \zeta = z + \lambda C,$$

whence $$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = \lambda^2 V^2 = \rho^2,$$

ρ being a distance measured along the normal; the equation of the plane in question is

$$(\xi - x - \tfrac{1}{2}dx - \tfrac{1}{4}d^2x - \&\text{c.})\ (dx + \tfrac{1}{2}d^2x + \&\text{c.}) + \ .\ .\ . = 0,$$

or, substituting for $\xi - x$, $\eta - y$, $\zeta - z$ the values $\dfrac{\rho A}{V}$, $\dfrac{\rho B}{V}$, $\dfrac{\rho C}{V}$, the equation, omitting higher infinitesimals, becomes

$$\frac{\rho}{V}\{A\,(dx + \tfrac{1}{2}d^2x) + B\,(dy + \tfrac{1}{2}d^2y) + C(dz + \tfrac{1}{2}d^2z)\} = \tfrac{1}{2}(dx^2 + dy^2 + dz^2)\ ;$$

which, observing that $Adx + Bdy + Cdz = 0$, is

$$\rho(Ad^2x + Bd^2y + Cd^2z) - V\ (dx^2 + dy^2 + dz^2) = 0,$$

or, substituting for dx, . . ., d^2x, . . . their values, it is

$$\rho\ (E'dp^2 + 2F'dpdq + G'dq^2) - V\ (Edp^2 + 2Fdpdq + Gdq^2) = 0,$$

the above-mentioned equation.*

The formula explains the meaning of the coefficients E', F', G'; it shows that the equation

$$E'dp^2 + 2F'dpdq + G'dq^2 = 0$$

determines the directions of the inflexional tangents at the point (p, q). It may be observed that if $E' = 0$, $G' = 0$, this

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* This equation is obtained geometrically by Williamson, *Quarterly Journal*, Vol. XI. p. 364 (1871).

equation becomes $dpdq=0$, we then have $p=$ const., $q=$ const., as the equations of the "inflexion curves," or curves which at each point thereof coincide in direction with an inflexional tangent.

382. We may imagine the parameters p, q so determined that the equations of the two sets of lines of curvature shall be $p=$ const. and $q=$ const. respectively. When this is so the differential equation of the lines of curvature will be $dpdq=0$; and this will be the case if $F=0$, $F'=0$; we thus obtain $F=0$, $F'=0$ as *the conditions in order that the equations of the lines of curvature may be* $p=$ *const.* and $q=$ *const.* Or, writing the conditions at full length, they are

$$\frac{dx}{dp}\frac{dx}{dq}+\frac{dy}{dp}\frac{dy}{dq}+\frac{dz}{dp}\frac{dz}{dq}=0,$$

$$\begin{vmatrix} \dfrac{dx}{dp}, & \dfrac{dy}{dp}, & \dfrac{dz}{dp} \\ \dfrac{dx}{dq}, & \dfrac{dy}{dq}, & \dfrac{dz}{dq} \\ \dfrac{d^2x}{dpdq}, & \dfrac{d^2y}{dpdq}, & \dfrac{d^2z}{dpdq} \end{vmatrix}=0,$$

where it may be noticed that the first equation merely expresses that the curves $p=$ const. and $q=$ const. intersect at right angles.

[The equation $F'=0$ expresses the condition that the parametric curves determine conjugate directions (Art. 268). For if in the limit the tangent plane at the point $x+\frac{dx}{dp}\delta p$, $y+\frac{dy}{dp}\delta p$, $z+\frac{dz}{dp}\delta p$ intersects the tangent plane at x, y, z in the line whose direction-cosines are proportional to $\frac{dx}{dq}$, $\frac{dy}{dq}$, $\frac{dz}{dq}$, we must have not only

$$A\frac{dx}{dq}+B\frac{dy}{dq}+C\frac{dz}{dq}=0,$$

but also $\left(A+\frac{dA}{dp}\delta p\right)\frac{dx}{dq}+\left(B+\frac{dB}{dp}\delta p\right)\frac{dy}{dq}+\left(C+\frac{dC}{dp}\delta p\right)\frac{dz}{dq}=0$

and therefore $a'\frac{dA}{dp}+b'\frac{dB}{dp}+c'\frac{dC}{dp}=0$,

and by differentiating the identity

$$Aa'+Bb'+Cc'=0$$

this gives $A\frac{da'}{dp}+B\frac{db'}{dp}+C\frac{dc'}{dp}$, or $F'=0$.]

383. If, as above, $F=0$, $F'=0$, then the quadratic equation for ρ is

$$(\rho E' - VE)\ (\rho G' - VG) = 0,$$

and from the equations of Art. 380, putting successively $dp=0$, $dq=0$, it appears that the value $\rho = \frac{VG}{G'}$ belongs to the line of curvature $p=$ const., and the value $\rho = \frac{VE}{E'}$ to the line of curvature $q=$ const.

384. The above determinant-equation $F'=0$ may be replaced by three equations

$$\frac{d^2x}{dpdq} + \lambda\frac{dx}{dp} + \mu\frac{dx}{dq} = 0, \text{ \&c.},$$

where λ, μ, are indeterminate coefficients; multiplying first by $\frac{dx}{dp}$, $\frac{dy}{dp}$, $\frac{dz}{dp}$, and adding, we have an equation containing only λ, and which is

$$\frac{1}{2}\frac{dE}{dq} + \lambda E = 0,$$

and similarly multiplying by $\frac{dx}{dq}$, $\frac{dy}{dq}$, $\frac{dz}{dq}$, and adding, we obtain

$$\frac{1}{2}\frac{dG}{dp} + \mu G = 0.$$

It thus appears, that $p=$ *const.*, $q=$ *const.*, *being the equations of the curves of curvature, the coordinates* x, y, z *considered as functions* of p, q *satisfy each the partial differential equation*

$$\frac{d^2u}{dpdq} - \frac{1}{2}\frac{1}{E}\frac{dE}{dq}\frac{du}{dp} - \frac{1}{2}\frac{1}{G}\frac{dG}{dp}\frac{du}{dq} = 0.^*$$

[Conversely, if $f_1(p, q)$, $f_2(p, q)$, $f_3(p, q)$ are linearly independent solutions of an equation of the type $\frac{d^2u}{dpdq} + \lambda\frac{du}{dp} + \mu\frac{du}{dq} = 0$, λ and μ being functions of p and q, then the equations $x = f_1(p, q)$, $y = f_2(p, q)$, $z = f_3(p, q)$ define a surface on which the parametric curves $p =$ const., $q =$ const., are conjugate systems (i.e. the tangents to the parametric curves at any point are conjugate directions as defined in Art. 268). If in addition $F = 0$, the parametric curves are lines of curvature.

* See Lamé, *Leçons sur les coordonnées curvilignes.* Paris, 1859, p. 89,

Darboux uses this method to prove that *lines of curvature on any surface invert into lines of curvature on the inverse surface.* The radius of inversion being unity, and $x'y'z'$ the inverse of a point x, y, z on the original surface, we may express x', y', z', as functions of p and q by the equations $x = x'\rho$, $y = y'\rho$, $z = z'\rho$, where $\rho = x^2 + y^2 + z^2$. It is easy to see that $F = 0$ implies that $\frac{dx'}{dp}\cdot\frac{dx'}{dq} + \frac{dy'}{dp}\cdot\frac{dy'}{dq} + \frac{dz'}{dp}\cdot\frac{dz'}{dq} = 0$. Now if x, y, z satisfy the differential equation $\frac{d^2u}{dpdq} + \lambda\frac{du}{dp} + \mu\frac{du}{dq} = 0$, by substituting $x = x'\rho$, $y = y'\rho$, $z = z'\rho$, we find that x', y', z' satisfy a differential equation of the form

$$\frac{d^2v}{dpdq} + \lambda'\frac{dv}{dp} + \mu'\frac{dv}{dq} = 0, \text{ provided } \frac{d^2\rho}{dpdq} + \lambda\frac{d\rho}{dp} + \mu\frac{d\rho}{dq} = 0.$$

But the latter condition is found to be satisfied, provided that $F = 0$. Eliminating λ', μ' from the three equations satisfied by x', y', z', we obtain

$$\begin{vmatrix} \frac{dx'}{dp} & \frac{dy'}{dp} & \frac{dz'}{dp} \\ \frac{dx'}{dq} & \frac{dy'}{dq} & \frac{dz'}{dq} \\ \frac{d^2x'}{dpdq} & \frac{d^2y'}{dpdq} & \frac{d^2z'}{dpdq} \end{vmatrix} = 0.]$$

385. Entering now upon Gauss's theory of the curvature of surfaces,* it is to be remembered that in plane curves we measure the curvature of an arc of given length by the angle between the tangents, or between the normals, at its extremities: in other words, if we take a circle whose radius is unity, and draw radii parallel to the normals at the extremities of the arc, the ratio of the intercepted arc of the circle to the arc of the curve affords a measure of the curvature of the arc. In like manner, if we have a portion of a surface bounded by any closed curve, and if we draw radii of a unit sphere parallel to the normals at every point of the bounding curve,† the area of the corresponding portion of the sphere is called by Gauss the *total curvature* of the portion of the surface under consideration. And if at any point of a surface we divide the total curvature of the superficial element adjacent to the point by the area of the element itself, the quotient is called the *measure of curvature* for that point.

* See his Memoir referred to in Note to Art. 377.

† [This mode of representing the points on a surface on a sphere is sometimes called the *spherical representation* of the surface.]

386 We proceed *to express the measure of curvature by a formula* Since the tangent planes at any point on the surface, and at the corresponding point on the unit sphere, are by hypothesis parallel, the areas of any elementary portions of each are proportional to their projections on any of the coordinate planes. Let us consider, then, their projections on the plane of xy, and let us suppose the equation of the surface to be given in the form $z = \phi\,(x, y)$

If then x, y, z be the coordinates of any point on the surface, X, Y, Z those of the corresponding point on the unit sphere, $x + dx$, $x + \delta x$, $X + dX$ $X + \delta X$, &c, the coordinates of two adjacent points on each, then the areas of the two elementary triangles formed by the points considered are evidently in the ratio

$$dX\delta Y - dY\delta X : dx\delta y - dy\delta x$$

But dX, dY, δX, δY are connected with dx, dy, &c., by the same linear transformations, viz

$$dX = \frac{dX}{dx}dx + \frac{dX}{dy}dy, \quad dY = \frac{dY}{dx}dx + \frac{dY}{dy}dy\ ;$$

$$\delta X = \frac{dX}{dx}\delta x + \frac{dX}{dy}\delta y, \quad \delta Y = \frac{dY}{dx}\delta x + \frac{dY}{dy}\delta y\ ;$$

whence, by the theory of linear transformations, or by actual multiplication,

$$dX\delta Y - dY\delta X = (dx\delta y - dy\delta x)\left(\frac{dX}{dx}\frac{dY}{dy} - \frac{dX}{dy}\frac{dY}{dx}\right),$$

thus the quantity $\dfrac{dX}{dx}\dfrac{dY}{dy} - \dfrac{dX}{dy}\dfrac{dY}{dx}$ is the measure of curvature

Now X, Y, Z, being the projections on the axes of a unit line parallel to the normal, are proportional to the cosines of the angles which the normal makes with the axes. We have, therefore,

$$X = \frac{p}{\sqrt{(1 + p^2 + q^2)}}, \quad Y = \frac{q}{\sqrt{(1 + p^2 + q^2)}},$$

$$\frac{dX}{dx} = \frac{(1 + q^2)r - pqs}{(1 + p^2 + q^2)^{\frac{3}{2}}}, \quad \frac{dX}{dy} = \frac{(1 + q^2)s - pqt}{(1 + p^2 + q^2)^{\frac{3}{2}}},$$

$$\frac{dY}{dx} = \frac{(1 + p^2)s - pqr}{(1 + p^2 + q^2)^{\frac{3}{2}}}, \quad \frac{dY}{dy} = \frac{(1 + p^2)t - pqs}{(1 + p^2 + q^2)^{\frac{3}{2}}},$$

whence
$$\frac{dX}{dx}\frac{dY}{dy} - \frac{dX}{dy}\frac{dY}{dx} = \frac{rt - s^2}{(1+p^2+q^2)^2}.$$

But from the equation of Art. 311, it appears that the value just found for *the measure of curvature is* $\frac{1}{RR'}$, *where* R *and* R' *are the two principal radii of curvature at the point.*

387. It is easy to verify geometrically the value thus found. For consider the elementary rectangle whose sides are in the directions of the principal tangents. Let the lengths of the sides be λ, λ', and consequently its area $\lambda\lambda'$. Now the normals at the extremities of λ intersect, and if they make with each other an angle θ, we have $\theta = \lambda : R$ where R is the corresponding radius of curvature. But the corresponding normals of the sphere make with each other, by hypothesis, the same angle, and their length is unity. Denoting, therefore, by μ the length of the element on the sphere corresponding to λ, we have $\frac{\lambda}{R} = \mu$. In like manner we have $\frac{\lambda'}{R'} = \mu$, and $\frac{\mu\mu'}{\lambda\lambda'} = \frac{1}{RR'}$, which was to be proved.

388. From the formula of Art. 379, it appears that *the value of the measure of curvature is*
$$= \frac{1}{(EG - F^2)^2}(E'G' - F'^2),$$
but Gauss obtains this expression in a very different form, as a function of only E, F, G, and their differential coefficients in regard to p, q. To obtain this result we have to express in this form the function $E'G' - F'^2$: that is, the function
$$\begin{vmatrix} a, \beta, \gamma \\ a, b, c \\ a', b', c' \end{vmatrix} \times \begin{vmatrix} a'', \beta'', \gamma'' \\ a, b, c \\ a', b', c' \end{vmatrix} - \begin{vmatrix} a', \beta', \gamma' \\ a, b, c \\ a', b', c' \end{vmatrix}^2.$$
Now if these products be expanded according to the ordinary rule for multiplication of determinants, they give the difference between the two determinants*

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* I owe to Mr. Williamson the remark, that the application of this rule exhibits the result in a form which manifests the truth of Gauss's theorem.

$$\begin{vmatrix} \alpha\alpha''+\beta\beta''+\gamma\gamma'', & a\alpha''+b\beta''+c\gamma'', & a'\alpha''+b'\beta''+c'\gamma'' \\ a\alpha+b\beta+c\gamma, & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha+b'\beta+c'\gamma, & aa'+bb'+cc', & a'^2+b'^2+c'^2 \end{vmatrix}$$

$$\begin{vmatrix} \alpha'^2+\beta'^2+\gamma'^2, & a\alpha'+b\beta'+c\gamma', & a'\alpha'+b'\beta'+c'\gamma' \\ a\alpha'+b\beta'+c\gamma', & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha'+b'\beta'+c'\gamma', & aa'+bb'+cc', & a'^2+b'^2+c'^2 \end{vmatrix}.$$

389. Now it is easy to show that *the terms in these determinants are functions of E, F, G and their differentials.* Referring to the definitions of a, b, c, α, α', α'', &c. (Art. 377) it is obvious that

$$\alpha=\frac{da}{dp},\ \alpha'=\frac{da}{dq}=\frac{da'}{dp},\ \alpha''=\frac{da'}{dq},\ \&c.,$$

whence, since

$$E=a^2+b^2+c^2,\ F=aa'+bb'+cc',\ G=a'^2+b'^2+c'^2,$$

$$a\alpha+b\beta+c\gamma=\tfrac{1}{2}\frac{dE}{dp},\quad a\alpha'+b\beta'+c\gamma'=\tfrac{1}{2}\frac{dE}{dq},$$

$$a'\alpha'+b'\beta'+c'\gamma'=\tfrac{1}{2}\frac{dG}{dp},\ a'\alpha''+b'\beta''+c'\gamma''=\tfrac{1}{2}\frac{dG}{dq},$$

$$a\alpha''+b\beta''+c\gamma''=\frac{dF}{dq}-(a'\alpha'+b'\beta'+c'\gamma')=\frac{dF}{dq}-\tfrac{1}{2}\frac{dG}{dp},$$

$$a'\alpha+b'\beta+c'\gamma=\frac{dF}{dp}-(a\alpha'+b\beta'+c\gamma')=\frac{dF}{dp}-\tfrac{1}{2}\frac{dE}{dq}.$$

It will be seen that these equations express in terms of E, F, G every term in the preceding determinants except the leading one in each. To express these, differentiate, with regard to q, the equation last written, and we have

$$\alpha\alpha''+\beta\beta''+\gamma\gamma''=\frac{d^2F}{dpdq}-\tfrac{1}{2}\frac{d^2E}{dq^2}-\left(a'\frac{d\alpha}{dq}+b'\frac{d\beta}{dq}+c'\frac{d\gamma}{dq}\right).$$

Again, differentiate, with regard to p, the equation

$$a'\alpha'+b'\beta'+c'\gamma'=\tfrac{1}{2}\frac{dG}{dp},$$

and we have

$$\alpha'^2+\beta'^2+\gamma'^2=\tfrac{1}{2}\frac{d^2G}{dp^2}-\left(a'\frac{d\alpha'}{dp}+b'\frac{d\beta'}{dp}+c'\frac{d\gamma'}{dp}\right).$$

Now because $\frac{da}{dq} = \frac{da'}{dp}$, &c., the quantities within the brackets in the last two equations are equal. And since the leading term in each determinant is multiplied by the same factor, in subtracting the determinants we are only concerned with the difference of these terms, and the quantity within the brackets disappears from the result. The function in question is thus equal to the difference of the determinants

$$\begin{vmatrix} \frac{d^2F}{dpdq} - \frac{1}{2}\frac{d^2E}{dq^2}, & \frac{dF}{dq} - \frac{1}{2}\frac{dG}{dp}, & \frac{1}{2}\frac{dG}{dq} \\ \frac{1}{2}\frac{dE}{dp}, & E, & F \\ \frac{dF}{dp} - \frac{1}{2}\frac{dE}{dq}, & F, & G \end{vmatrix},$$

and

$$\begin{vmatrix} \frac{1}{2}\frac{d^2G}{dp^2}, & \frac{1}{2}\frac{dE}{dq}, & \frac{1}{2}\frac{dG}{dp} \\ \frac{1}{2}\frac{dE}{dq}, & E, & F \\ \frac{1}{2}\frac{dG}{dp}, & F, & G \end{vmatrix}.$$

We get the measure of curvature by dividing the quantity now found by $(EG - F^2)^2$, and the result is thus a function of E, F, G and their differentials. Gauss's theorem is therefore proved. It may be remarked that the expression involves only second differential coefficients of E, F, G, that is third differential coefficients of the coordinates; these, however, really disappear, since the original expression $E'G' - F'^2$ involves only second differential coefficients of the coordinates.

We add the actual expansion of the determinants, though not necessary to the proof. Writing the measure of curvature k, we have

$$4(EG - F^2)^2k = E\left\{\frac{dE}{dq}\frac{dG}{dq} - 2\frac{dF}{dp}\frac{dG}{dq} + \left(\frac{dG}{dp}\right)^2\right\}$$
$$+ F\left\{\frac{dE}{dp}\frac{dG}{dq} - \frac{dE}{dq}\frac{dG}{dp} - 2\frac{dE}{dq}\frac{dF}{dq} + 4\frac{dF}{dp}\frac{dF}{dq} - 2\frac{dF}{dp}\frac{dG}{dp}\right\}$$

$$+ G\left\{\frac{dE}{dp}\frac{dG}{dp} - 2\frac{dE}{dp}\frac{dF}{dq} + \left(\frac{dE}{dq}\right)^2\right\}$$
$$- 2(EG - F^2)\left(\frac{d^2E}{dq^2} - 2\frac{d^2F}{dpdq} + \frac{d^2G}{dp^2}\right),$$

(Liouville's Monge, p. 523).*

[Ex. 1. A system of confocals may be represented by

$$\frac{x^2}{\alpha - t} + \frac{y^2}{\beta - t} + \frac{z^2}{\gamma - t} = 1.$$

Let λ, μ, ν be the three values of t corresponding to the three confocals through a point. μ and ν being the parameters on the surface $\lambda = \lambda_0$, find the values of E, F, G, E', F', G'.

Prove that $F = 0$, $F' = 0$, and interpret the meaning of these identities.

Ex. 2. Find the values, in the same case, of the principal radii and of the measure of curvature (on $\lambda = \lambda_0$) in terms of μ, ν, and verify the results by Art. 197.]

390. The foregoing theorem, that the measure of curvature is a function of E, F, G and their differentials, shows that *if a surface supposed to be flexible, but not extensible, be transformed in any manner; that is to say, if the shape of the surface be changed, yet so that the distance between any two points measured along the surface remains the same, then the measure of curvature at every point remains unaltered.* We have an example of this change in the case of a developable surface which is such a deformation of a plane; and the measure of curvature vanishes for the developable, as well as for the plane, one of the principal radii being infinite. To see that the general theorem is true, observe that the expression of an element of length on the surface is

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

Let x', y', z' denote the point of the deformed surface corresponding to any point x, y, z of the original surface. Then x', y', z' are given functions of x, y, z, and can therefore also

* Bertrand, Diguet, and Puiseux (see *Liouville*, Vol. XIII. p. 80; Appendix to Monge, p. 583) have established Gauss's theorem by calculating the perimeter and area of a goedesic circle on any surface, whose radius, supposed to be very small, is s. They find for the perimeter $2\pi s - \frac{\pi s^3}{3RR'}$, and for the area $\pi s^2 - \frac{\pi s^4}{12RR'}$. And of course the supposition that these are unaltered by deformation implies that RR' is constant.

be expressed in terms of p, q; and the element of any arc of the deformed surface can be expressed in the form

$$ds'^2 = E_1 dp^2 + 2F_1 dpdq + G_1 dq^2.$$

But *the condition that the length of the arc shall be unaltered by transformation, manifestly requires that* $E = E_1$, $F = F_1$, $G = G_1$; hence, *any function of* E, F, G, *and their differentials with regard to* p *and* q *and, in particular the value of the measure of curvature, is unaltered by the deformation in question.*

391. We may consider two systems of curves traced on the surface, for one of which p is constant, and for the other q; so that any point on the surface is the intersection of a pair of curves, one belonging to each system. The expression then $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$ shows that $\sqrt{(E)}\,dp$ is the element of the curve, passing through the point, for which q is constant; and $\sqrt{(G)}\,dq$ is the element of the curve for which p is constant. If these two curves intersect at an angle ω, then since ds is the diagonal of a parallelogram of which $\sqrt{(E)}dp$, $\sqrt{(G)}dq$ are the sides, we have $\sqrt{(EG)}\cos\omega = F$, while the area of the parallelogram is $d\sigma d\sigma' \sin\omega = \sqrt{(EG - F^2)}\,dpdq$.

If the curves of the system p cut at right angles those of the system q, we must have $F = 0$.

A particular case of these formulæ is when we use geodesic polar coordinates, in which case, as we shall subsequently show, we always have an expression of the form $ds^2 = d\rho^2 + P^2 d\omega^2$. Now if in the formula of Art. 389 we put $F = 0$, $E =$ constant, it becomes

$$4E^2G^2k = E\left(\frac{dG}{dp}\right)^2 - 2EG\,\frac{d^2G}{dp^2},$$

and if we put

$$E = 1,\ G = P^2,\ p = \rho,\ k = \frac{1}{RR'},\ \text{we have}\ \frac{d^2P}{d\rho^2} + \frac{P}{RR'} = 0,$$

an equation which must be satisfied by the function P on any surface, if $Pd\omega$ expresses the element of the arc of a geodesic circle. Roberts verifies (*Cambridge and Dublin Mathematical Journal*, Vol. III. p. 161) that this equation is satisfied by the function y cosec ω on a quadric.

[Ex. 1. If θ be the angle, at their point of intersection, between two curves C_1 and C_2 lying on a surface prove

$$\cos\theta = \frac{E dp_1 dp_2 + F(dp_1 dq_2 + dq_1 dp_2) + G(dq_1 dq_2)}{ds_1 ds_2},$$

$$\sin\theta = \pm V\left(\frac{dp_1}{ds_1}\frac{dq_2}{ds_2} - \frac{dp_2}{ds_2}\frac{dq_1}{ds_1}\right).$$

Ex. 2. The orthogonal trajectories of the family of curves $Mdp + Ndq = 0$ are the family represented by $(FM - EN)\,dp + (GM - FN)\,dq = 0$.

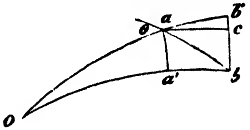
Ex. 3. Two directions are conjugate if $E'\,dp_1\,dp_2 + F'\,(dp_1\,dq_2 + dq_1\,dp_2) + G'\,dq_1\,dq_2 = 0$; and the family conjugate to the family $Mdp + Ndq = 0$ is

$$(F'M - E'N)\,dp - (F'N - G'M)\,dq = 0.$$

(Two singly infinite families of curves on a surface are said to be conjugate when the directions of the tangent lines at each point are conjugate, Art. 268).]

392. Gauss applies these formulæ to find the *total curvature, in his sense of the word, of a geodesic triangle* on any surface. The element of the area being $Pd\omega d\rho$, and the measure of curvature being $-\frac{1}{P}\frac{d^2P}{d\rho^2}$, by twice integrating $-\frac{d^2P}{d^2\rho}d\rho d\omega$ the total curvature is found. Integrating first with respect to ρ, we get $\left(C - \frac{dP}{d\rho}\right)d\omega$. Now if the radii are measured from one vertex of the given triangle, the integral is plainly to vanish for $\rho = 0$; and it is plain also that for $\rho = 0$ we must have $\frac{dP}{d\rho} = 1$; for as ρ tends to vanish, the length of an element perpendicular to the radius tends to become $\rho d\omega$. Hence the first integral is $d\omega\left(1 - \frac{dP}{d\rho}\right)$.

This may be written in a more convenient form as follows: Let θ be the angle which any radius vector makes with the element of a geodesic arc ab. Now since $aa' = Pd\omega$, $bb' = (P + dP)\,d\omega$; and if $cb = aa'$, we have $cb' = dPd\omega$, and the angle $cab' = \frac{dP}{d\rho}d\omega$. But cab' is evidently the diminution



of the angle θ in passing to a consecutive point; hence $d\theta = -\frac{dP}{d\rho}d\omega$. The integral just found is therefore $d\omega + d\theta$, which integrated a second time is $\omega + \theta' - \theta''$, where ω is the angle between the two extreme radii vectores which we consider, and θ', θ'' are the corresponding values of θ. If we call A, B, C the internal angles of the triangle formed by the two extreme radii and by the base, we have $\omega = A$, $\theta' = B$, $\theta'' = \pi - C$, and the total curvature is $A + B + C - \pi$. Hence *the excess over* 180° *of the sum of the angles of a geodesic triangle is measured by the area of that portion of the unit sphere which corresponds to the direction of the normals along the sides of the given triangle.*

The portion on the unit sphere corresponding to the area enclosed by a geodesic returning upon itself is half the sphere. For if the radius vector travel round so as to return to the point whence it set out, the extreme values of θ' and θ'' are equal, while ω has increased by 2π. The measure of curvature is therefore 2π, or half the surface of the sphere.

Gauss elsewhere applies the formulæ to the representation of one surface on another, and in particular to the representation of a surface on a plane, in such manner that the infinitesimal elements of the one surface are similar to those of the other; a condition satisfied in the stereographic projection and in other representations of the sphere.*

[392*a*. **Conformal Representation.**—The "representation" of one class of entities on another signifies, in general, a one-to-one correspondence between the entities. Projection (Art. 144*c*) is an example of the representation of one space on another or on itself. The deformation, without tearing, of a surface (Art. 390) is an example of the representation of one surface on another. Representation when the angles between corresponding curves at their points of

* For some other interesting theorems, relative to the deformation of surfaces see Jellett's paper "On the Properties of Inextensible Surfaces," *Transactions of the Royal Irish Academy*, Vol. XXII. Memoirs have also appeared by Bour and Bonnet, on the theory of Surfaces applicable to one another, to one of which was awarded the prize of the French Academy in 1860. [The literature of this subject is very large; see, for example, the treatises of Darboux (1896) or Bianchi (1902-1909).]

intersection is the same, is said to be *conformal*. If A_1, B_1, C_1 and A_2, B_2, C_2 are corresponding points on two surfaces S_1 and S_2, then we assume that, as B_1, C_1 approach towards coincidence with A_1; B_2, C_2 approach towards coincidence with A_2, and the limiting values of the angles of the triangle A_1, B_1, C_1 are equal to the limiting values of the angles of the triangle A_2, B_2, C_2. All corresponding triangles in the neighbourhood of corresponding points P_1 and P_2 are therefore similar to the first order of small quantities. Hence $\frac{ds_1}{ds_2}$ is a function of the position of the point P_1 (or P_2), where $\frac{ds_1}{ds_2}$ represents the limiting value of the ratio of corresponding arcs drawn from P_1 and P_2. We may suppose as in Art. 390, that the parametric curves p and q correspond; therefore if

$$ds_1^2 = E_1dp^2 + 2F_1dpdq + G_1dq^2,$$

and

$$ds_2^2 = E_2dp^2 + 2F_2dpdq + G_2dq^2,$$

the *necessary and sufficient condition for conformal representation is*

$$\frac{E_2}{E_1} = \frac{F_2}{F_1} = \frac{G_2}{G_1} = \lambda^2, *$$

where λ is a function of p, q. λ is the linear magnification of infinitesimal representation and λ^2 the areal magnification.

Let p, q represent an orthogonal system ($F_1 = 0$) on the surface S_1, then they correspond to an orthogonal system on $S_2(F_2 = 0)$. Hence

$$ds_1^2 = E_1dp^2 + G_1dq^2, \text{ and } ds_2^2 = E_2dp^2 + G_2dq^2.$$

Now functions α_1, β_1, of p and q exist such that

$$\mu_1(\sqrt{E_1}dp + i\sqrt{G_1}dq) = d\alpha_1,\ \nu_1(\sqrt{E_1}dp - i\sqrt{G_1}dq) = d\beta_1,$$

where $i = \sqrt{-1}$, and μ_1, ν_1 are functions of p and q. Then $ds_1^2 - t_1d\alpha_1d\beta_1$ where $t_1\mu_1\nu_1 = 1$. The imaginary lines on S_1 represented by $\alpha_1 = \text{const.}$, $\beta_1 = \text{const.}$ are called *minimal* lines. There are two and two only through each point, their directions being determined for general parameters u, v by $Edu^2 + 2Fdudv + Gdv^2 = 0$. Now let α_2, β_2 be minimal lines on S_2, then $ds_2^2 = t_2d\alpha_2d\beta_2$. Hence the condition for conformal representation gives $\frac{t_2d\alpha_2d\beta_2}{t_1d\alpha_1d\beta_1} = \lambda^2$. Hence if the direction determined on S_1 by a given value of $dp : dq$ satisfies $d\alpha_1 = 0$, the corresponding direction on S_2 (for which $dp : dq$ has the same value) must satisfy either $d\alpha_2 = 0$ or $d\beta_2 = 0$; similarly if $d\beta_1 = 0$ either $d\alpha_2$ or $d\beta_2 = 0$. The necessary and sufficient condition for this is clearly $\alpha_2 = f(\alpha_1)$, $\beta_2 = g(\beta_1)$ or $\alpha_2 = f(\beta_1)$, $\beta_2 = g(\alpha_1)$, f and g being arbitrary functions, which are conjugate imaginaries if the surfaces are real. This may be expressed by saying that *the necessary and sufficient condition that a representation be conformal is that the minimal lines on the two surfaces correspond.*

In general we do not know the parameters p, q which determine the correspondence, and the problem is given the parametric lines p_1, q_1, on S_1 to find the corresponding parametric lines p_2, q_2, on S_2, i.e. to express p_2, q_2 as

* This may be proved analytically by using Ex. 1, Art. 391.

functions of p_1, q_1. We may suppose that both parametric lines are orthogonal and therefore $ds_1^2 = E_1dp_1^2 + G_1dq_1^2$, $ds_2^2 = E_2dp_2^2 + G_2dq_2^2$. We determine the minimal lines as before and it is easy to see that α_1, β_1 may be expressed as conjugate imaginary functions of p_1, q_1. Thus

$$\alpha_1 = \phi_1(p_1, q_1) + i\psi_1(p_1, q_1),\ \beta_1 = \phi_1(p_1, q_1) - i\psi_1(p_1, q_1)$$
$$\alpha_2 = \phi_2(p_2, q_2) + i\psi_2(p_2, q_2),\ \beta_2 = \phi_2(p_2, q_2) - i\psi_2(p_2, q_2).$$

We may now replace the parameters (p_1, q_1), (p_2, q_2) by (ϕ_1, ψ_1), (ϕ_2, ψ_2), and then the conformal representation is given by

$$\phi_2 + i\psi_2 = f(\phi_1 + i\psi_1), \text{ or by } \phi_2 + i\psi_2 = f(\phi_1 - i\psi_1).$$

It may be proved that the two representations have opposite *sense*, illustrated by the difference *in plano* between inversion and *homothety* (or the correspondence of similar figures that may be similarly placed by translation and rotation).

We have $ds_1^2 = \lambda_1(d\phi_1^2 + d\psi_1^2)$, and $ds_2^2 = \lambda_2(d\phi_2^2 + d\psi_2^2)$. Both sets of parametric lines are then *isothermal* and the parameters are *isometric* (Ex. 2, Art. 396*a*). It is easy to see that ϕ_1, ψ_1 may be *any* isometric parameters on S_1, and ϕ_2, ψ_2 any isometric parameters on S_2, for the only condition is that the minimal lines correspond and the minimal lines for $ds^2 = \lambda(du^2 + dv^2)$ form only two families which are given by $u + iv = \text{const.}$, $u - iv = \text{const.}$* Therefore *the most general conformal representation of one surface on another is given by an arbitrary functional relation between the complex variables (or their conjugates) of the isometric parameters on the two surfaces.*

As an example, consider the *conformal representation of a sphere on a plane.* The radius being unity $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$ and

$$ds^2 = \sin^2\theta\left\{\left(\frac{d\theta}{\sin\theta}\right)^2 + (d\phi)^2\right\} = \lambda^2(d\phi^2 + d\psi^2) \text{ where } \psi = \log\left(\tan\frac{\theta}{2}\right).$$

Thus ϕ, ψ are isometric parameters, and the general representation is given by $\phi + i\psi = f(x \pm iy)$, x, y being rectangular coordinates in the plane.

The simplest case is

$$x = \phi,\ y = \psi = \log\left(\tan\frac{\theta}{2}\right).$$

This is *Mercator's chart.* The paths corresponding to the lines $x = \text{const.}$ on the chart are meridians and any path corresponding to a straight line on the chart is a *loxodrome*, i.e. it cuts the meridians at a constant angle.]

393. It remains to say something of the properties of curves considered as belonging to a particular surface. Thus the sphere we know has a geometry of its own, where great circles take the place of lines in a plane; and, in like manner,

* By considering the minimal lines it may easily be proved that any two real isometric systems, (u, v) and (u', v') are connected by equations of the form $u' + iv' = f(u + iv)$, or $u' + iv' = g(u - iv)$.

each surface has a geometry of its own, the geodesics on that surface answering to right lines.*

We have already by anticipation given the fundamental property of a *geodesic* (Art. 308). The *differential equation* is immediately obtained from the property there proved, that the normal lies in the plane of two successive elements of the curve and bisects the angle between them; *hence L, M, N, which are proportional to the direction-cosines of the normal, must be proportional to* $d\frac{dx}{ds}$, $d\frac{dy}{ds}$, $d\frac{dz}{ds}$, which are the direction-cosines of the bisector (Art. 358).

Thus "if the tangents to a geodesic make a constant angle with a fixed plane, the normals along it will be parallel to that plane, and *vice versâ* (Dickson, *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 168). For from the equation

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = \text{constant},$$

which denotes that the tangents make a constant angle with a fixed plane, we can deduce

$$aL + bM + cN = 0,$$

which denotes that the normals are parallel to the same plane.

394. *If through any point on a surface there be drawn two indefinitely near and equal geodesics, the line joining their extremities is at right angles to both.*†

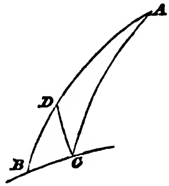
* The geometry of curves traced upon the hyperboloid of one sheet has been studied nearly in the same manner by Plucker, *Crelle*, Vol. XLIII. (1847), and by Chasles (*Comptes rendus*, Vol. LIII. 1861, p. 985), the coordinates made use of being the intercepts made by the two generators through any point on two fixed generators taken for axes. It is easy to show that in this method the most general equation of a plane section is of the form

$$Axy + Bx + Cy + D = 0,$$

and generally that the order of any curve is equal to the sum of the highest powers of x and y in its equation, whether these highest powers occur in the same term or not. The curves are distinguished into families according to the number of intersections of the curve by the generating lines of the two kinds respectively. Thus, for a quartic curve of the first kind, or quadriquadric, each generating line of either kind meets the curve in 2 points; but for a quartic curve of the second kind, or excubo-quartic, each generating line of the one kind meets the curve in 3 points, and each generating line of the other kind in one point.

† This theorem is due to Gauss, who also proves it by the Calculus of Variations; see the Appendix to Liouville's Edition of Monge, p. 528.

Let $AB = AC$, and let us suppose the angle at B not to be right, but to be $= \theta$. Take $BD = BC \sec\theta$, and then, because all the sides of the triangle BCD are infinitely small, it may be treated as a plane triangle and the angle DCB is a right angle. We have therefore $DC < DB$, $AD + DC < AB$, and therefore $< AC$. It follows that AC is not the shortest path from A to C, contrary to hypothesis. Or the proof may be stated thus: The shortest line from a point A to any curve on a surface meets that curve perpendicularly. For if not, take a point D on the radius vector from A and indefinitely near to the curve; and from this point let fall a perpendicular on the curve, which we can do by taking along BC a portion $BD \cos\theta$ and joining the point so found to D. We can pass then from D to the curve more shortly by going along the perpendicular than by travelling along the assumed radius vector, which is therefore not the shortest path.



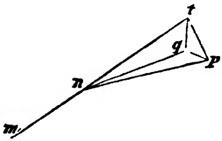
Hence, *if every geodesic through A meet the curve perpendicularly, the length of that geodesic is constant.* It is also evident, mechanically, that the curve described on any surface by a strained cord from a fixed point is everywhere perpendicular to the direction of the cord.

[The formula $ds^2 = d\rho^2 + P^2 d\omega^2$ (Art. 391) follows from the theorem of this Article.]

395. The theorem just proved is the fundamental theorem of the method of infinitesimals, applied to right lines (*Conics*, pp. 369, &c.). All the theorems therefore which are there proved by means of this principle will be true if instead of right lines we consider geodesics traced on any surface. For example, "if we construct on any surface the curve answering to an ellipse or hyperbola; that is to say, the locus of a point the sum or difference of whose geodesic distances from two fixed points on the surface is constant; then the tangent at any point of the locus bisects the angle between the

geodesics joining the point of contact to the fixed points." The converse of this theorem is also true. Again, "if two geodesic tangents to a curve, through any point P, make equal angles with the tangent to a curve along which P moves, then the difference between the sum of these tangents and the intercepted arc of the curve which they touch is constant" (see *Conics*, Art. 399). Again, "if equal portions be taken on the geodesic normals to a curve, the line joining their extremities cuts all at right angles," or, "if two different curves both cut at right angles a system of geodesics they intercept a constant length on each vector of the series." We shall presently apply these principles to the case of geodesics traced on quadrics.

396. As the curvature of a plane curve is measured by the ratio which the angle between two consecutive tangents bears to the elements of the arc, so the *geodesic curvature* of a curve on a surface is measured by the ratio borne to the element of the arc by the angle between two consecutive geodesic tangents. The following calculation of the radius of geodesic curvature, due to Liouville,* gives at the same time a proof of Meunier's theorem.



Let mn, np be two consecutive and equal elements of the curve. Produce $nt = mn$, and let fall tq perpendicular to the surface; join nq and qp. Then, since nt makes an infinitely small angle with the surface, its projection nq is equal to it. nq is the second element of the normal section and is also the second element of the geodesic production of mn. If now θ be the angle of contact tnp, and θ' be tnq the angle of contact of the normal section, we have $tp = \theta ds$, $tq = \theta' ds$. Now the angle qtp $(= \phi)$ is the angle between the osculating plane of the curve and the plane of normal section, and since $tq = tp \cos\phi$, we have $\theta' = \theta \cos\phi$ and $\frac{1}{R} = \frac{\cos\phi}{\rho}$, which is

* Appendix to Monge, p. 576.

Meunier's theorem; R being the radius of curvature of the normal section and ρ that of the given curve.

Now, in like manner, pnq being θ'' the geodesic angle of contact, we have $pq = \theta'' ds$ and $pq = tp \sin\phi$ or $\frac{1}{r} = \frac{\sin\phi}{\rho}$. The *geodesic* radius of curvature is therefore* ρ cosec ϕ. It is easy to see that this *geodesic radius is the absolute radius of curvature of the plane curve into which the given curve would be transformed, by circumscribing a developable to the given surface along the given curve, and unfolding that developable into a plane. The geodesic curvature at a point may also be defined as the curvature, at the point, of the orthogonal projection of the curve on the tangent plane.*

[396*a*. *The geodesic curvature of the parametric curves can be expressed in terms of E, F, G, and their differentials with regard to p and q.* Let $p = P$, $q = Q$ be the parametric curves at a point, P and Q being constant. Consider the curve $p = P$, and let its elements be symbolised as in Art. 368*a*, and those of the curve $q = Q$ by the corresponding accented letters. Let ϕ be the angle between the principal normal and the normal to the surface, $\frac{1}{r}$ the geodesic curvature, θ the angle between the parametric curves, ψ the angle between the principal normal to $p = P$ and the tangent line to $q = Q$. Then by spherical trigonometry $\cos\psi = \sin\phi \sin\theta$ Hence (Art. 391) $\sin\phi = \frac{\sqrt{EG}}{V}\cos\psi$.

Now $\cos\psi = l\alpha' + m\beta' + n\gamma'$, and (Art. 391) $ds' = \sqrt{E}\,dp$, therefore

$$\sqrt{E}\cos\psi = l\frac{\delta x}{\delta p} + m\frac{\delta y}{\delta p} + n\frac{\delta z}{\delta p},$$

δ denoting partial differentiation.

Using Frenet's formulas

$$\frac{l}{\rho} = \frac{d\alpha}{ds} = \frac{\delta\alpha}{\delta q}\frac{dq}{ds}, \text{ also } \alpha = \frac{1}{\sqrt{G}}\frac{\delta x}{\delta q} \text{ and } ds = \sqrt{G}\,dq.$$

Making these substitutions and remembering $\frac{1}{r} = \frac{\sin\phi}{\rho}$,

we find

$$\frac{V}{r} = \frac{1}{\sqrt{G}}\Sigma\frac{\delta x}{\delta p}\frac{\delta^2 x}{\delta q^2} + \frac{\delta}{\delta q}\left(\frac{1}{\sqrt{G}}\right)\Sigma\frac{\delta x}{\delta p}\frac{\delta x}{\delta q}.$$

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* I have not adopted the name "second geodesic curvature" introduced by Bonnet. It is intended to express the ratio borne to the element of the arc by the angle which the normal at one extremity makes with the plane containing the element and the normal at the other extremity.

Now (Art. 377) $E = \Sigma\left(\frac{\delta x}{\delta p}\right)^2$, $F = \Sigma\frac{\delta x}{\delta p}\cdot\frac{\delta x}{\delta q}$, $G = \Sigma\left(\frac{\delta x}{\delta q}\right)^2$,

therefore $\Sigma\frac{\delta x}{\delta p}\cdot\frac{\delta^2 x}{\delta^2 q} = \frac{\delta F}{\delta q} - \frac{1}{2}\frac{\delta G}{\delta p}$ and finally

$$\frac{1}{r} = \frac{1}{V}\left\{\frac{\delta}{\delta q}\left(\frac{F}{\sqrt{G}}\right) - \frac{\delta}{\delta p}(\sqrt{G})\right\}.$$

If the surface be deformed (as in Art. 390) we may suppose that a point whose curvilinear coordinates are p, q on the original surface is transposed into a point whose curvilinear coordinates on the new surface are also p, q. Hence the parametric curves $p = P$, $q = Q$ are deformed into the parametric curves $p = P$, $q = Q$. The values of E, F, G at corresponding points are the same, and they must evidently be the same functions of p and q. Therefore —since the parametric curve may be chosen arbitrarily—*if a surface be deformed without stretching, the geodesic curvature at each point of any curve on the surface remains unaltered.* In particular *geodesics are deformed into geodesics*, since $\frac{1}{r} = 0$.

Ex. 1. *If the parametric curves are at right angles* $F = 0$, and *the curves* $p =$ *const. will be geodesics if G is a function of q alone.*

Ex. 2. When the parametric curves are at right angles and have constant geodesic curvature, it is possible to find a system of coordinates with the same parametric curves so that $E = G = [\phi(p) + \psi(q)]^{-2}$.

Hence $ds^2 = E(dp^2 + dq^2)$. Such a system, where E may be *any* function of p and q, is said to be *isothermal* or *isometric*, since the curves $p =$ const. $q =$ const. divide the surface into a network of small squares. (Cf. Art. 392*a*.)

From the preceding formula for the geodesic curvature of any curve of the family $dp = 0$ we can find the *geodesic curvature of any curve of the family* $Mdp + Ndq = 0$, where M and N are functions of p and q. We have only to transform so that $u =$ const. will be parametric lines, where $Mdp + Ndq = tdu$. The other parametric lines, say $v =$ const., may be chosen arbitrarily; we may choose v, for example, so that u and v form an orthogonal system. In that case $Pdp + Qdq = t'dv$, where $P = FM - EN$, $Q = GM - FN$. Solve for dp and dq in terms of du and dv and equate the two expressions for ds^2, namely, $Edp^2 + 2Fdpdq + Gdq^2$ and $E_1du^2 + G_1dv^2$. Hence

$$TG_1 = t'^2,\ TE_1 = t^2V^2, \text{ where } T = EN^2 - 2FMN + GM^2.$$

Now the geodesic curvature for $u =$ constant, v being the other parametric line, is $-\frac{1}{\sqrt{E_1G_1}}\frac{\delta}{\delta u}(\sqrt{G_1})$, since $F_1 = 0$. Using $dp = \lambda du + \mu dv$, $dq = \lambda' du + \mu' dv$ we have $\frac{\delta p}{\delta u} = \lambda$, $\frac{\delta q}{\delta u} = \lambda'$, λ and λ' being known. Also $\frac{\delta}{\delta u} = \frac{\delta p}{\delta u}\cdot\frac{\delta}{\delta p} + \frac{\delta q}{\delta u}\cdot\frac{\delta}{\delta q}$.

The quantities t, t' will be found to be eliminated, if we express the condition that they are the reciprocals of the integrating factors of $Mdp + Ndq$, $Pdp + Qdq$. We get finally for the geodesic curvature of $Mdp + Ndq$

$$\frac{1}{V}\left\{\frac{\delta}{\delta q}\left(\frac{P}{T^{\frac{1}{2}}}\right) - \frac{\delta}{\delta p}\left(\frac{Q}{T^{\frac{1}{2}}}\right)\right\}.$$

By equating this expression to zero we obtain the condition that a given curve should be a geodesic. Another form of this condition may be derived from the equations

$$\Sigma\frac{\delta x}{\delta p}\frac{d^2x}{ds^2} = 0, \qquad \Sigma\frac{\delta x}{\delta q}\frac{d^2x}{ds^2} = 0,$$

which express that the principal normal to the curve is normal to the surface. These lead to two equations involving the first and second derivatives of p and q with respect to the arc s of the curve, the coefficients being functions of E, F, G and their first partial derivatives. The two equations are not independent, for either may be deduced from the other by means of the formula which expresses that ds is the element of arc. When $ds^2 = d\rho^2 + P^2d\omega^2$, these equations are

$$\frac{d^2\rho}{ds^2} - P\frac{\delta P}{\delta\rho}\left(\frac{d\omega}{ds}\right)^2 = 0, \qquad P\frac{d^2\omega}{ds^2} + 2\frac{\delta P}{\delta\rho}\frac{d\rho}{ds}\frac{d\omega}{ds} + \frac{\delta P}{\delta\omega}\left(\frac{d\omega}{ds}\right)^2 = 0.$$

Ex. 3. If the curve is given in the form $q = f(p)$ the condition that it be a geodesic may be expressed as a differential equation of the form

$$f'' + af'^3 + bf'^2 + cf' + d = 0,$$

where $f' = \frac{dq}{dp}$, $f'' = \frac{d^2q}{dp^2}$, and a, b, c, d, are functions involving E, F, G, and their differentials with regard to p and q (Bianchi, *Lezioni*, Art. 87).

396*b*. When a curve lies on a surface, its *geodesic torsion* at any point is defined as the torsion thereat of the geodesic touching the curve at the point. Bonnet has proved that its value is $\frac{1}{\tau} - \frac{d\phi}{ds}$, where τ is the torsion of the curve, and ϕ is the angle between the principal normal and the normal to the surface.

Let X, Y, Z, be the direction cosines of the normal to the surface; then

$$\cos\phi = lX + mY + nZ$$
$$\sin\phi = \lambda X + \mu Y + \nu Z.$$

Differentiating the latter equation and using the Frenet-Serret formulas we find

$$\cos\phi\left(\frac{d\phi}{ds} - \frac{1}{\tau}\right) = \lambda\frac{dX}{ds} + \mu\frac{dY}{ds} + \nu\frac{dZ}{ds}.$$

Take for axes of x, y, z, the directions of principal curvature and the normal to the surface; then the equation of the surface is

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} + \text{higher powers}.$$

Let θ be the angle between the tangent to the curve and the axis of x; then $\alpha = \cos\theta$, $\beta = \sin\theta$, $\gamma = 0$; $n = \cos\phi$; and since (paying attention to the signs adopted, Art. 368*a*) $\lambda = \beta n - \gamma m$, $\mu = \gamma l - \alpha n$, we have also $\qquad \lambda = \sin\theta\cos\phi$, $\mu = -\cos\theta\cos\phi$

$$\text{Again, } X = \frac{-p}{\sqrt{1+p^2+q^2}},\ Y = \frac{-q}{\sqrt{1+p^2+q^2}},\ Z = \frac{1}{\sqrt{1+p^2+q^2}}$$

where $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$. Since $\alpha = \frac{dx}{ds}$, $\beta = \frac{dy}{ds}$, $\gamma = \frac{dz}{ds}$, we have, at the

origin, $\frac{dX}{ds} = -\frac{\alpha}{\rho_1}$, $\frac{dY}{ds} = -\frac{\beta}{\rho_2}$, $\frac{dZ}{ds} = 0$, also $X = Y = 0$, $Z = 1$. Making these substitutions, and dividing across by $\cos\phi$,

$$\frac{1}{\tau} - \frac{d\phi}{ds} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\theta\cos\theta,$$

and has therefore the same value for all curves at the point having the same tangent line.

For a geodesic ϕ is zero; hence the *geodesic torsion for the direction θ is equal to* $\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\theta\cos\theta$.

An *asymptotic line* has to be considered separately. Differentiating with regard to s the equation $p\alpha + q\beta - \gamma = 0$, and using $\tan\theta = \sqrt{-\frac{\rho_2}{\rho_1}}$ with the Frenet-Serret formulas, we find $\frac{pl + qm - n}{\rho} + \frac{\alpha^2}{\rho_1} + \frac{\beta^2}{\rho_2} = 0$; hence $pl + qm - n = 0$. Therefore *the osculating plane is the tangent plane to the surface*, and $\cos\phi = 0$. Thus the above proof appears to fail since we divided across by $\cos\phi$. But the formula is still true. For the torsion of any curve is $l\frac{d\lambda}{ds} + m\frac{d\mu}{ds} + n\frac{d\nu}{ds}$. Now for an asymptotic line $\lambda = \pm X$, $\mu = \pm Y$, $\nu = \pm Z$, and if we make these substitutions and put $l = \beta Z - \gamma Y$, the torsion is $\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\theta\cos\theta$. Substituting $\tan\theta = \pm\sqrt{-\frac{\rho_2}{\rho_1}}$ we find $\frac{1}{\tau} = \sqrt{-\frac{1}{\rho_1\rho_2}}$, that is, *the square of the torsion of an asymptotic line is equal to the measure of curvature with changed sign*, and *the torsion of the two asymptotic lines differ only in sign*. (Enneper.)

Cor. 1. *The geodesic torsion of a line of curvature is zero*, or if a geodesic touch a line of curvature, it has a stationary plane at the point. This is the same as Lancret's theorem (Art. 312)

Cor. 2. A geodesic passing through an umbilic has a stationary plane thereat.

Ex. 1. At any point on a geodesic

$$\frac{\cos\theta}{\rho} + \frac{\sin\theta}{\tau} = \frac{\cos\theta}{\rho_1}$$
$$\frac{\sin\theta}{\rho} - \frac{\cos\theta}{\tau} = \frac{\sin\theta}{\rho_2}.$$

Ex. 2. If two surfaces cut at a constant angle, their curve of intersection is a line of curvature on both or on neither; and if their curve of intersection is a line of curvature of both, the surfaces cut at a constant angle (Joachimsthal). This theorem, proved otherwise in Art. 304, also follows from Bonnet's formula.

Ex. 3. *To find the geodesic torsion of a curve defined by $f(p, q) = 0$.* It is equal to $\frac{-\Delta}{V^2(Edp^2 + 2Fdpdq + Gdq^2)}$, where Δ is the determinant of Art. 378,

$dp : dq$ satisfying $\frac{\delta f}{\delta p} dp + \frac{\delta f}{\delta q} dq = 0$. For $\cos\phi$ multiplied by the geodesic torsion $= -\Sigma\lambda \frac{dX}{ds} =$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ X & Y & Z \\ X' & Y' & Z' \end{vmatrix} \begin{vmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ X & Y & Z \end{vmatrix}$$

X, Y, Z being the direction-cosines of the normal to the surface, and therefore equal to $\frac{A}{V}, \frac{B}{V}, \frac{C}{V}$. But the second determinant is $-\cos\phi$, therefore the first is equal to the geodesic torsion with its sign changed. Also

$$\Delta = V^2 ds^3 \begin{vmatrix} \alpha & \beta & \gamma \\ X & Y & Z \\ X' & Y' & Z' \end{vmatrix}$$

where $X' = \frac{dX}{ds}$, &c., and the result follows, since $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$. Finally Δ is expressed in terms of dp, dq, as in Art. 378.]

397. The theory of *geodesics traced on quadrics* depends on Jacobi's first integral of the differential equation of these lines; intimately connected herewith we have Joachimsthal's fundamental theorem, that *at every point on such a curve pD is constant*, where, as at Art. 166, p is the perpendicular from the centre on the tangent plane at the point, and D is the diameter of the quadric parallel to the tangent to the curve at the same point. This may be proved by the help of the two following principles: (1) If from any point two tangent lines be drawn to a quadric, their lengths are proportional to the parallel diameters. This is evident from Art. 74; and (2) If from each of two points A, B on the quadric perpendiculars be let fall on the tangent plane at the other, these perpendiculars will be proportional to the perpendiculars from the centre on the same planes. For the length of the perpendicular from $x''y''z''$ on the tangent plane at $x'y'z'$ is $p\left(\frac{x'x''}{a^2} + \frac{y'y''}{b^2} + \frac{z'z''}{c^2} - 1\right)$, and the perpendicular from $x'y'z'$ on the tangent plane at $x''y''z''$ is $p'\left(\frac{x'x''}{a^2} + \frac{y'y''}{b^2} + \frac{z'z''}{c^2} - 1\right)$.

If now from the points A, B there be drawn lines AT, BT to any point T on the intersection of the tangent planes at A and B, and if AT make an angle i with the intersection of the

planes, the angle between the planes being ω, then the perpendicular from A to the intersection of the planes is $AT \sin i$, and from A on the other plane is $AT \sin i \sin \omega$. In like manner the perpendicular from B on the tangent plane at A is $BT \sin i' \sin \omega$. If, therefore, the lines AT, BT make equal angles with the intersection of the planes, the lines AT, BT are proportional to the perpendiculars from A and B on the two planes. But AT and BT are proportional to D and D', and the perpendiculars are as the perpendiculars from the centre p' and p. Hence $Dp = D'p'$. But it was proved (Art. 308) that if AT, TB be successive elements of a geodesic, they make equal angles with the intersection of the tangent planes at A and B. Hence, the quantity pD remains unchanged as we pass from point to point of the geodesic. Q.E.D.*

398. On account of the importance of the preceding theorem we wish also to show how it may be deduced from the differential equations of a geodesic.† Differentiating the equation

$$\frac{L^2}{R^2} + \frac{M^2}{R^2} + \frac{N^2}{R^2} = 1$$

(where L, M, N are the differential coefficients and

$$R^2 = L^2 + M^2 + N^2),$$

and then substituting for L, &c., $d\frac{dx}{ds}$, &c. (Art. 393), we get

$$d\left(\frac{dx}{ds}\right) d\left(\frac{L}{R}\right) + d\left(\frac{dy}{ds}\right) d\left(\frac{M}{R}\right) + d\left(\frac{dz}{ds}\right) d\left(\frac{N}{R}\right) = 0.$$

It is to be remarked, that this equation is also true for a line of curvature; for since $L : R$, &c. are the direction-cosines of the normal, the direction-cosines of a line in the same

* This proof is by Graves, *Crelle*, Vol. XLII. p. 279.

† See Jacobi, *Crelle*, Vol. XIX. (1839), p. 309; Joachimsthal, *Crelle*, Vol. XXVI. p. 155; Bonnet, *Journal de l'Ecole Polytechnique*, Vol. XIX. p. 138; Dickson, *Cambridge and Dublin Mathematical Journal*, Vol. V. p. 168; Jacobi, *Vorlesungen über Dynamik*, p. 212. The theory of geodesic lines on a spheroid of revolution, in particular an oblate spheroid, was considered by Legendre.

plane with two consecutive normals, and perpendicular to them, are (Art. 358) proportional to $d\left(\frac{L}{R}\right)$, &c. Hence the $\frac{dx}{ds}$, &c. of a line of curvature are proportional to $d\left(\frac{L}{R}\right)$, &c. But if now we differentiate

$$\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}+\frac{dz^2}{ds^2}=1,$$

and substitute for $\frac{dx}{ds}$, &c. the values just given, we have again the equation

$$d\left(\frac{dx}{ds}\right)d\left(\frac{L}{R}\right)+d\left(\frac{dy}{ds}\right)d\left(\frac{M}{R}\right)+d\left(\frac{dz}{ds}\right)d\left(\frac{N}{R}\right)=0.$$

If we actually perform the differentiations, and reduce the result by the differential equation of the surface $Ldx+Mdy+Ndz=0$, and its consequence

$$dLdx+dMdy+dNdz=-(Ld^2x+Md^2y+Nd^2z),$$

we get

$$(dLdx+dMdy+dNdz)\ (dRds-Rd^2s)$$
$$+(dLd^2x+dMd^2y+dNd^2z)Rds=0,\text{*}$$

or
$$\frac{dLd^2x+dMd^2y+dNd^2z}{dLdx+dMdy+dNdz}+\frac{dR}{R}-\frac{d^2s}{ds}=0.$$

399. The preceding equation is true for a geodesic or for

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* Gehring has remarked (see Hesse, *Vorlesungen*, p. 325) that this equation multiplied by Rds, subject as before to the condition $Ldx+Mdy+Ndz=0$, may be resolved into the product of the two determinants

$$\begin{vmatrix} dx, & dy, & dz \\ L, & M, & N \\ dL, & dM, & dN \end{vmatrix} \cdot \begin{vmatrix} dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \\ L, & M, & N \end{vmatrix}.$$

So that *for quadrics the determinant of the lines of curvature is the integrating factor of the geodesics.* Hesse shows that the integral so arrived at belongs exclusively to the latter.

[It may be noticed that, for a curve on the surface whose elements are dx, dy, dz, and s is the independent variable, the first of these determinants is $-\frac{R^2ds^3}{\tau}$ where $\frac{1}{\tau}$ is the geodesic torsion (Art. 396); and the second determinant is $\frac{R\sin\omega\, ds^3}{\rho}$ where ω is the angle between the principal normal to the curve and the normal to the surface.]

a line of curvature on any surface, but *when the surface is only of the second degree, a first integral of the equation can be found.* In fact we have

$$dLd^2x + dMd^2y + dNd^2z = \tfrac{1}{2}d\,(dLdx + dMdy + dNdz).$$

This may be easily verified by using the general equation of a quadric, or, more simply, by using the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

when $\quad L = \dfrac{x}{a^2},\ M = \dfrac{y}{b^2},\ N = \dfrac{z}{c^2};\ dL = \dfrac{dx}{a^2},\ dM = \dfrac{dy}{b^2},\ dN = \dfrac{dz}{c^2};$

by substituting which values the equation is at once established.

The equation of the last article then consists of terms, each separately integrable. Integrating, we have

$$R^2(dLdx + dMdy + dNdz) = Cds^2.$$

Now, from the preceding values,

$$R^2 = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{1}{p^2},$$

and $\quad \dfrac{dL}{ds}\dfrac{dx}{ds} + \dfrac{dM}{ds}\dfrac{dy}{ds} + \dfrac{dN}{ds}\dfrac{dz}{ds} = \dfrac{1}{a^2}\dfrac{dx^2}{ds^2} + \dfrac{1}{b^2}\dfrac{dy^2}{ds^2} + \dfrac{1}{c^2}\dfrac{dz^2}{ds^2}.$

But the right-hand side of the equation denotes the reciprocal of the square of a central radius whose direction-cosines are $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}.$

*The geometric meaning therefore of the integral we have found is pD = constant.**

400. *The constant pD has the same value for all geodesics which pass through an umbilic.*

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* Hart proves the same theorem as follows. Consider any plane section of an ellipsoid, let ϖ be the perpendicular from the centre of the section on the tangent line, d the diameter of the section parallel to that tangent, i the angle the plane of the section makes with the tangent plane at any point. Then along the section ϖd is constant, and it is evident that pD is in a fixed ratio to $\varpi d \sin i$. Hence along the section pD varies as $\sin i$ and will be a maximum where the plane meets the surface perpendicularly. But a geodesic osculates a series of normal sections; therefore, for such a line pD is constant, its differential always vanishing. *Cambridge and Dublin Mathematical Journal*, Vol. IV. p. 84.

For at the umbilic the p is of course common to all, being $= ac : b$ and, since the central section parallel to the tangent plane at the umbilic is a circle, the diameter parallel to the tangent line to the geodesic is constant, being always equal to the mean axis b. Hence, for a geodesic passing through an umbilic we have $pD = ac$.

Let now any point on a quadric be joined by geodesics to two umbilics, since we have just proved that pD is the same for both geodesics, and, since at the point of meeting the p is the same for both, the D for that point must also have the same value for both; that is to say, the diameters are equal which are drawn parallel to the tangents to the geodesics at their point of meeting. But two equal diameters of a conic make equal angles with its axes; and we know that the axes of the central section of a quadric parallel to the tangent plane at any point are parallel to the directions of the lines of curvature at that point. Hence, *the geodesics joining any point on a quadric to two umbilics make equal angles with the lines of curvature through that point.**

It follows that the geodesics joining any point to the two opposite umbilics, which lie on the same diameter, are continuations of each other, since the vertically opposite angles are equal which these geodesics make with either line of curvature through the point.

It follows also (see Art. 395) that *the sum or difference is constant of the geodesic distances of all the points on the same line of curvature from two umbilics.* The sum is constant when the two umbilics chosen are interior with respect to the line of curvature; the difference, when for one of these umbilics we substitute that diametrically opposite, so that one of the umbilics is interior, the other exterior to the line of curvature.

If A, A' be two opposite umbilics, and B another umbilic, since the sum $PA + PB$ is constant, and also the difference

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* This theorem and its consequences developed in the following articles are due to Michael Roberts, *Liouville*, Vol. XI. p. 1.

$PA' - PB$, it follows that $PA + PA'$ is constant; that is to say, *all the geodesics which connect two opposite umbilics are of equal length.* In fact, it is evident that two indefinitely near geodesics connecting the same two points on any surface must be equal to each other.

401. *The constant pD has the same value for all geodesics which touch the same line of curvature.*

It was proved (Art. 166) that pD has a constant value all along a line of curvature; but at the points where either geodesic touches the line of curvature both p and D have the same value for the geodesic and the line of curvature.

Hence, then, a *system of lines of curvature has properties completely analogous to those of a system of confocal conics in a plane; the umbilics answering to the foci.* For example, *two geodesic tangents drawn to one from any point on another make equal angles with the tangent at that point.* Graves's theorem for plane conics holds also for lines of curvature, viz. that the excess of the sum of two tangents to a line of curvature over the intercepted arc is constant, while the intersection moves along another line of curvature of the same species (see *Conics*, Art. 399).

[The constancy of pD for a geodesic and for a line of curvature is a particular case of the following general theorem. *For any curve on a quadric, the product of the geodesic torsion and the geodesic curvature is equal to*

$$\frac{p^3}{2}\frac{d}{ds}\left(\frac{1}{pD}\right)^2.$$

This may be proved by using the method of Art. 399, combined with the result stated in the conclusion of Note, Art. 398.]

402. The equation $pD = \text{constant}$ has been written in another convenient form.* Let a', a'' be the primary semi-axes of two confocal surfaces through any point on the curve, and let i be the angle which the tangent to the geodesic makes with one of the principal tangents. Then, since $a^2 - a'^2$, $a^2 - a''^2$ (Art. 164) are the semi-axes of the central section

* By *Liouville*, Vol. IX. p. 401.

parallel to the tangent plane, any other semi-diameter of that section is given by the equation

$$\frac{1}{D^2}=\frac{\cos^2 i}{a^2-a''^2}+\frac{\sin^2 i}{a^2-a'^2},$$

while, again, $\quad \dfrac{1}{p^2}=\dfrac{(a^2-a'^2)(a^2-a''^2)}{a^2b^2c^2}$ (Art. 165).

The equation, therefore, $pD=$ constant is equivalent to

$$(a^2-a'^2)\cos^2 i+(a^2-a''^2)\sin^2 i=\text{constant},$$

or to

$$a'^2\cos^2 i+a''^2\sin^2 i=\text{constant}.$$

403. *The locus of the intersection of two geodesic tangents to a line of curvature, which cut at right angles, is a sphero-conic.*

This is proved as the corresponding theorem for plane conics. If a', a'' belong to the point of intersection, we have

$$a'^2\cos^2 i+a''^2\sin^2 i=\text{constant},\ a'^2\sin^2 i+a''^2\cos^2 i=\text{constant},$$

hence

$$a'^2+a''^2=\text{constant}:$$

and therefore (Art. 161) the distance of the point of intersection from the centre of the quadric is constant. The locus of intersection is therefore the intersection of the given quadric with a concentric sphere. The demonstration holds if the geodesics are tangents to different lines of curvature; and, as a particular case, the locus of the foot of the geodesic perpendicular from an umbilic on the tangent to a line of curvature is a sphero-conic.

404. *To find the locus of intersection of geodesic tangents to a line of curvature which cut at a given angle* (Besge, *Liouville*, Vol. XIV. p. 247).

The tangents from any point whose a', a'' are given, to a given line of curvature, are determined by the equation $a'^2\cos^2 i+a''^2\sin^2 i=\beta$; and since they make equal angles with either of the principal tangents through that point, i the angle they make with one of these tangents is half the angle they make with each other. We have therefore

$$\tan\tfrac{1}{2}\theta=\frac{\sqrt{(\beta-a''^2)}}{\sqrt{(a'^2-\beta)}};\ \tan\theta=\frac{2\sqrt{(\beta-a''^2)}\sqrt{(a'^2-\beta)}}{a'^2+a''^2-2\beta},$$

$$(a'^2+a''^2-2\beta)^2\tan^2\theta=4\beta(a'^2+a''^2)-4a'^2a''^2-4\beta^2.$$

This is reduced to ordinary coordinates by the equations (Arts 160, 161)

$$a'^2 + a''^2 = x^2 + y^2 + z^2 - b^2 - c^2 + a^2, \quad a'^2 a''^2 = \frac{x^2(a^2-b^2)(a^2-c^2)}{a^2},$$

whence it appears that the locus required is the intersection of the quadric with a surface of the fourth degree *

405 It was proved (Art 176) that two confocals can be drawn to touch a given line, that if the axes of the three surfaces passing through any point on the line be a, a', a'', and the angles the line makes with the three normals at the point be α, β, γ, then the axis-major of the touched confocal is determined by the quadratic

$$\frac{\cos^2\alpha}{a^2 - \mathbf{a}^2} + \frac{\cos^2\beta}{a'^2 - \mathbf{a}^2} + \frac{\cos^2\gamma}{a''^2 - \mathbf{a}^2} = 0$$

Let us suppose now that the given line is a tangent to the quadric whose axis is a, we have then $\cos\alpha = 0$, since the line is of course at right angles to the normal to the first surface, and we have $\cos\beta = \sin\gamma$, since the tangent plane to the surface a contains both the line and the other two normals The angle γ is what we have called i in the articles immediately preceding The axis then of the second confocal touched by the given line is determined by the equation

$$\frac{\sin^2 i}{a'^2 - \mathbf{a}^2} + \frac{\cos^2 i}{a''^2 - \mathbf{a}^2} = 0, \text{ or } a'^2 \cos^2 i + a''^2 \sin^2 i = \mathbf{a}^2$$

If, then, we write the equation of a geodesic (Art 402) $a'^2 \cos^2 i + a''^2 \sin^2 i = \mathbf{a}^2$, we see from this article that that equation expresses that *all the tangent lines along the same geodesic touch the confocal surface whose primary axis is* $\mathbf{a}$ †

The geodesic itself will touch the line of curvature in which this confocal intersects the original surface, for the

* Michael Roberts has proved (*Liouville*, Vol XV. p. 291) by the method of Art 188, that the projection of this curve on the plane of circular sections is the locus of the intersection of tangents, cutting at a constant angle, to the conic into which the line of curvature is projected

† The theorems of this article are taken from Chasles's Memoir, *Liouville*, Vol XI p 5

tangent to the geodesic at the point where the geodesic meets the confocal is, as we have just proved, also the tangent to the confocal at that point. The geodesic, therefore, and the intersection of the confocal with the given surface have a common tangent.

The osculating planes of the geodesic are obviously tangent planes to the same confocal, since they are the planes of two consecutive tangent lines to that confocal.

The value of pD for a geodesic passing through an umbilic is ac (Art. 400); and the corresponding equation is, therefore, $a'^2\cos^2 i + a''^2\sin^2 i = a^2 - b^2$. Now the confocal, whose primary axis is $\sqrt{(a^2-b^2)}$, reduces to the umbilicar focal conic. Hence, as a particular case of the theorems just proved, *all tangent lines to a geodesic which passes through an umbilic intersect the umbilicar focal conic.*

Conversely, if from any point O on that focal conic rectilinear tangents be drawn to a quadric, and those tangents produced geodetically on the surface, the lines so produced will pass through the opposite umbilic; the whole lengths from O to the umbilic being equal.

406. From the fact (proved Art. 176) that tangent planes drawn through any line to the two confocals which touch it are at right angles to each other, we might have inferred directly, precisely as at Art. 309, that tangent lines to a geodesic touch a confocal. For the plane of two consecutive tangents to a geodesic being normal to the surface is tangent to the confocal touched by the first tangent. The second tangent to the geodesic, therefore, touches the same confocal; as, in like manner, do all the succeeding tangents. Having thus established the theorem of the last article, we could, by reversing the steps of the proof, obtain an independent demonstration of the theorem pD = constant.

407. *The developable circumscribed to a quadric along a geodesic has its cuspidal edge on another quadric, which is the same for all the geodesics touching the same line of curvature.*

For any point on the cuspidal edge is the intersection of three consecutive tangent planes to the given quadric, and the three points of contact, by hypothesis, determine an osculating plane of a geodesic which (Art. 405) touches a fixed confocal. The point on the cuspidal edge is the pole of this plane with respect to the given quadric; but the pole with respect to one quadric of a tangent plane to another lies on a third fixed quadric.

[Ex. If to a line of curvature, $a'^2 =$ const., on a given quadric (a, b, c), geodesic tangents be drawn and developables circumscribed to the quadric along each geodesic, prove that the locus of cuspidal edges on all such geodesics is the quadric

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = (a^2 - a'^2)\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right).]$$

408. Chasles has given the following generalization of Roberts's theorem, Art. 400. *If a thread fastened at two fixed points on one quadric A be strained by a pencil moving along a confocal B* (so that the thread of course lies in geodesics where it is in contact with the quadrics and in right lines in the space between them), *then the pencil will trace a line of curvature on the quadric B.* For the two geodesics on the surface B, which meet in the locus point P, evidently make equal angles with the locus of P; but these geodesics have, as tangents, the rectilinear parts of the thread which both touch the same confocal; therefore (Art. 405) the pD is the same for both geodesics, and hence the line bisecting the angle between them is a line of curvature.

A particular case of this theorem is, that the focal ellipse of a quadric can be described by means of a thread fastened to two fixed points on opposite branches of the focal hyperbola (*cf.* Art. 185).

409. *Elliptic Coordinates.* The method used (Arts. 403-4) in which the position of a point on the ellipsoid is defined by the primary axes of the two hyperboloids intersecting in that point, is called the method of Elliptic Coordinates (see Art. 188). As it is more convenient to work

with unaccented letters, I follow Liouville* in denoting the quantities which we have hitherto called a', a'' by the letters μ, ν; and in this notation the equations of the lines of curvature of one system are of the form $\mu =$ constant, and those of the other $\nu =$ constant. The equation of a geodesic (Art. 402) becomes

$$\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2;$$

and when the geodesic passes through an umbilic, we have $\mu'^2 = a^2 - b^2 = h^2$. It will be remembered (Arts. 159, 160) that μ lies between the limits k and h, and ν between the limits h and 0.

Throwing the equation of a geodesic into the form

$$\mu^2 + \nu^2 \tan^2 i = \mu'^2 (1 + \tan^2 i),$$

we see that it is satisfied (whatever be μ') by the values $\mu^2 = \nu^2$, $\tan^2 i = -1$. Hence it follows, that the same pair of imaginary tangents, drawn from an umbilic, touch all the lines of curvature,† a further analogy to the foci of plane conics.

410. *To express in elliptic coordinates the element of the arc of any curve on the surface.* Let us consider, first, the element of any line of curvature, $\mu =$ constant. Let that line

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* This method is evidently a particular case of that explained, Art. 377. In Cayley's Memoir on Geodesics (*Proceedings of London Mathematical Society*, 1872, p. 199) he uses the coordinates in a slightly different form; viz. if any point on the quadric $\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c} = 1$ is the intersection with it of the two confocals

$$\frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} = 1, \quad \frac{x^2}{a+q} + \frac{y^2}{b+q} + \frac{z^2}{c+q} = 1;$$

then p and q are the two coordinates: $p =$ const., $q =$ const. denote lines of curvature; and we have, by Art. 160, expressions for x, y, z in terms of p and q. The differential equation of the right lines of the surface is

$$\frac{dp}{\sqrt{\{(a+p)(b+p)(c+p)\}}} \pm \frac{dq}{\sqrt{\{(a+q)(b+q)(c+q)\}}} = 0.$$

In the ordinary case where the surface is an ellipsoid and $a > b > c$, the coordinates p and q may be distinguished by supposing p to range between the limits $-a$, $-b$, and q between $-b$, $-c$.

† Roberts, *Liouville*, Vol. XV. p. 289.

be met by the two consecutive hyperboloids, whose axes are ν and $\nu+d\nu$; then, since it cuts them perpendicularly, the intercept between them is equal to the difference between the central perpendiculars on parallel tangent planes to the two hyperboloids. But (Art. 180), $(p''+dp'')^2-p''^2=(\nu+d\nu)^2-\nu^2$ or $p''dp''=\nu d\nu$. Now we have proved that $dp''=d\sigma$, the element of the arc we are seeking, and

$$p''^2=\frac{a''^2b''^2c''^2}{(a^2-a''^2)\ (a'^2-a''^2)}=\frac{\nu^2(h^2-\nu^2)\ (k^2-\nu^2)}{(a^2-\nu^2)\ (\mu^2-\nu^2)}.$$

Hence $$d\sigma^2=\frac{(a^2-\nu^2)\ (\mu^2-\nu^2)}{(h^2-\nu^2)\ (k^2-\nu^2)}d\nu^2.$$

In like manner, the element of the arc of the line of curvature $\nu=$ constant is given by the formula

$$d\sigma'^2=\frac{(a^2-\mu^2)\ (\mu^2-\nu^2)}{(\mu^2-h^2)\ (k^2-\mu^2)}d\mu^2.$$

Now, if through the extremities of the element of the arc ds of any curve we draw lines of curvature of both systems, we form an elementary rectangle of which $d\sigma$, $d\sigma'$ are the sides and ds the diagonal. Hence

$$ds^2=\frac{(a^2-\mu^2)\ (\mu^2-\nu^2)}{(\mu^2-h^2)\ (k^2-\mu^2)}d\mu^2+\frac{(a^2-\nu^2)\ (\mu^2-\nu^2)}{(h^2-\nu^2)\ (k^2-\nu^2)}d\nu^2.$$

411. In like manner we can express *the area of any portion of the surface* bounded by four lines of curvature; two lines μ_1, μ_2 and two ν_1, ν_2. For the element of the area is

$$d\sigma_1 d\sigma_2=\frac{(\mu^2-\nu^2)\ \sqrt{\{(a^2-\mu^2)\ (a^2-\nu^2)\}}}{\sqrt{\{(\mu^2-h^2)\ (k^2-\mu^2)\ (h^2-\nu^2)\ (k^2-\nu^2)\}}}d\mu d\nu,$$

the integral of which is

$$\int_{\mu_2}^{\mu_1}\frac{\mu^2\sqrt{(a^2-\mu^2)}d\mu}{\sqrt{\{(\mu^2-h^2)\ (k^2-\mu^2)\}}}\int_{\nu_2}^{\nu_1}\frac{\sqrt{(a^2-\nu^2)}d\nu}{\sqrt{\{(h^2-\nu^2)\ (k^2-\nu^2)\}}}$$
$$-\int_{\mu_2}^{\mu_1}\frac{\sqrt{(a^2-\mu^2)}d\mu}{\sqrt{\{(\mu^2-h^2)\ (k^2-\mu^2)\}}}\int_{\nu_2}^{\nu_1}\frac{\nu^2\sqrt{(a^2-\nu^2)}d\nu}{\sqrt{\{(h^2-\nu^2)\ (k^2-\nu^2)\}}}.*$$

So, in like manner, we can find the *differential equation of the orthogonal trajectory of a curve* whose differential equation is $Md\mu+Nd\nu=0$. For the orthogonal trajectory to $Pd\sigma+Qd\sigma'$

* The area of the surface of the ellipsoid was thus first expressed by Legendre, *Traité des Fonctions Elliptiques*, Vol. I. p. 352.

is plainly $\frac{d\sigma}{P} = \frac{d\sigma'}{Q}$; since $d\sigma$, $d\sigma'$ are a system of rectangular coordinates. But $Md\mu + Nd\nu$ can be thrown without difficulty into the form $Pd\sigma + Qd\sigma'$ by the equations of the last article. The equation of the orthogonal trajectory is thus found to be

$$\frac{a^2 - \mu^2}{(\mu^2 - h^2)(k^2 - \mu^2)} \frac{d\mu}{M} - \frac{a^2 - \nu^2}{(h^2 - \nu^2)(k^2 - \nu^2)} \frac{d\nu}{N} = 0.$$

412. The first integral of a geodesic $\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2$ can be thrown into a form in which the variables are separated, and the *second integral can be obtained.* That equation gives

$$\tan i = \sqrt{\left(\frac{\mu^2 - \mu'^2}{\mu'^2 - \nu^2}\right)}.$$

But $$\tan i = \frac{d\sigma'}{d\sigma} = \frac{\sqrt{\{(a^2 - \mu^2)(h^2 - \nu^2)(k^2 - \nu^2)\}}}{\sqrt{\{(a^2 - \nu^2)(\mu^2 - h^2)(k^2 - \mu^2)\}}} \frac{d\mu}{d\nu},$$

whence, equating, we have

$$\frac{\sqrt{(a^2 - \mu^2)}\, d\mu}{\sqrt{(\mu^2 - \mu'^2)(\mu^2 - h^2)(k^2 - \mu^2)}} \pm \frac{\sqrt{(a^2 - \nu^2)}\, d\nu}{\sqrt{\{(\mu'^2 - \nu^2)(h^2 - \nu^2)(k^2 - \nu^2)\}}} = 0,$$

the terms of which can be integrated separately.*

If the geodesic passes through the umbilics, we have $\mu'^2 = h^2$ (Art. 409), and the equation of the geodesic is

$$\frac{\sqrt{(a^2 - \mu^2)}}{(\mu^2 - h^2)\sqrt{(k^2 - \mu^2)}} d\mu \pm \frac{\sqrt{(a^2 - \nu^2)}}{(h^2 - \nu^2)\sqrt{(k^2 - \nu^2)}} d\nu = 0.$$

413. *To find an expression for the length of any portion of a geodesic.* The element of the geodesic is the hypotenuse of a right-angled triangle, of which $d\sigma$, $d\sigma'$ are the sides, and whose

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* This is equivalent to Jacobi's first integral of the differential equation of the geodesic lines, see Art. 397; see also Hesse, *Vorlesungen*, p. 328. The reader is recommended also to refer to the method of integration employed by Weierstrass, *Monatsberichte der Berliner Akademie*, 1861, p. 986. The above equation in the notation used by Cayley is

$$dp\sqrt{\left\{\frac{p}{(a+p)(b+p)(c+p)(\theta+p)}\right\}} \pm dq\sqrt{\left\{\frac{q}{(a+q)(b+q)(c+q)(\theta+q)}\right\}} = 0,$$

where θ is the constant of integration. This is nearly the form given by Jacobi in the *Vorlesungen über Dynamik*, referred to in note to Art. 398.

base angle is i. Hence we have $ds = \sin i d\sigma' \pm \cos i d\sigma$; and putting in $\sin i = \frac{\sqrt{(\mu^2 - \mu'^2)}}{\sqrt{(\mu^2 - \nu^2)}}$, $\cos i = \frac{\sqrt{(\mu'^2 - \nu^2)}}{\sqrt{(\mu^2 - \nu^2)}}$, and giving $d\sigma$, $d\sigma'$ the values of Art. 410, we have

$$ds = d\mu\sqrt{\left\{\frac{(\mu^2 - \mu'^2)}{(\mu^2 - h^2)}\frac{(a^2 - \mu^2)}{(k^2 - \mu^2)}\right\}} \pm d\nu\sqrt{\left\{\frac{(\mu'^2 - \nu^2)}{(h^2 - \nu^2)}\frac{(a^2 - \nu^2)}{(k^2 - \nu^2)}\right\}}.$$

If ρ be the element of a line through the umbilics, we have

$$d\rho = d\mu\sqrt{\left(\frac{a^2 - \mu^2}{k^2 - \mu^2}\right)} \pm d\nu\sqrt{\left(\frac{a^2 - \nu^2}{k^2 - \nu^2}\right)}.$$

It is to be noted, that when we give to the radical in the last article the sign $+$, we must give that in this article the sign $-$. This appears by forming (Art. 411) the differential equation of the orthogonal trajectory to a geodesic through an umbilic, an equation which must be equivalent to $d\rho = 0$ (Art. 394).

414. In place of denoting the position of any point on an ellipsoid by the elliptic coordinates μ, ν, we might use *geodesic polar coordinates having the pole at an umbilic*, and denote a point by ρ, its geodesic distance from an umbilic, and by ω, the angle which that radius vector makes with the line joining the umbilics. Now the equation (Art. 412) of a geodesic passing through an umbilic gives the sum of two integrals equal to a constant. This constant cannot be a function of ρ, since it remains the same as we go along the same geodesic: it must therefore be a function of ω only; and if we pass from any point to an indefinitely near one, *not* on the same geodesic radius vector, we shall have

$$\frac{\sqrt{(a^2 - \mu^2)}d\mu}{(\mu^2 - h^2)\sqrt{(k^2 - \mu^2)}} \pm \frac{\sqrt{(a^2 - \nu^2)}d\nu}{(h^2 - \nu^2)\sqrt{(k^2 - \nu^2)}} = \phi'(\omega)d\omega.$$

We shall determine the form of the function by calculating its value for a point indefinitely near the umbilic, for $\mu = \nu = h$. The limit of the left-hand side of the equation then becomes

$\sqrt{\left(\frac{a^2 - h^2}{k^2 - h^2}\right)} \times$ limit of $\left(\frac{d\mu}{\mu^2 - h^2} + \frac{d\nu}{h^2 - \nu^2}\right)$. Now, if we put $\mu = h + \eta$, $\nu = h - \epsilon$, the quantity whose limit we want to find

is $\dfrac{d\eta}{2h\eta+\eta^2}-\dfrac{d\epsilon}{2h\epsilon-\epsilon^2}$, which, as η and ϵ tend to vanish, becomes the limit of $\dfrac{1}{2h}\left(\dfrac{d\eta}{\eta}-\dfrac{d\epsilon}{\epsilon}\right)$ or of $\dfrac{1}{2h}d\log\dfrac{\eta}{\epsilon}$.

Now since the angle external to the vertical angle of the triangle formed by the lines joining any point to two umbilics is bisected by the direction of the line of curvature, that external angle is double the angle i in the formula $\mu^2\cos^2 i+\nu^2\sin^2 i=h^2$. In the limit when the vertex of the triangle approaches the umbilic, the external angle of the triangle becomes ω, and we have at the umbilic

$$(h+\eta)^2\cos^2\tfrac{1}{2}\omega+(h-\epsilon)^2\sin^2\tfrac{1}{2}\omega=h^2,$$

and in the limit $\qquad \tan^2\tfrac{1}{2}\omega=\dfrac{\eta}{\epsilon}.$

Using this value, the limit of the left-hand side of the equation is

$$\frac{1}{2h}\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)}d\,(\log\tan^2\tfrac{1}{2}\omega).$$

We have therefore

$$\frac{\sqrt{(a^2-\mu^2)}d\mu}{(\mu^2-h^2)\sqrt{(k^2-\mu^2)}}+\frac{\sqrt{(a^2-\nu^2)}d\nu}{(h^2-\nu^2)\sqrt{(k^2-\nu^2)}}=\frac{1}{h}\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)}\frac{d\omega}{\sin\omega}.$$

And the constant which occurs in the integrated equation of a geodesic through an umbilic is of the form

$$\frac{1}{2h}\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)}\log\tan^2\tfrac{1}{2}\omega+C.$$

415. If P, Q be two consecutive points on a curve, and if PP' be drawn perpendicular to the geodesic radius vector OQ, it is evident that $PQ^2=PP'^2+P'Q^2$. Now since (Art. 394) $OP=OP'$ we have $P'Q=d\rho$, while PP' being the element of an arc of a geodesic circle, for which ρ is constant (or $d\rho=0$), must be of the form $Pd\omega$. Hence the element of the arc of a curve on any surface can be expressed by a formula $ds^2=d\rho^2+P^2d\omega^2$. We propose now to examine the form of the function P for the case of radii vectores drawn through an umbilic of an ellipsoid. Let us consider the line of curvature $\mu=\mu'$. We have then (Art. 413)

$$ds^2 = d\nu^2 \frac{(\mu'^2 - \nu^2)\ (a^2 - \nu^2)}{(h^2 - \nu^2)\ (k^2 - \nu^2)}.$$

And by the same article

$$d\rho^2 = d\nu^2 \frac{a^2 - \nu^2}{k^2 - \nu^2},$$

whence $$P^2 d\omega^2 = \frac{(\mu'^2 - h^2)\ (a^2 - \nu^2)}{(h^2 - \nu^2)\ (k^2 - \nu^2)} d\nu^2.$$

But (Art. 414), when μ is constant,

$$\frac{\sqrt{(a^2 - \nu^2)}d\nu}{(h^2 - \nu^2)\sqrt{(k^2 - \nu^2)}} = \frac{1}{h}\sqrt{\left(\frac{a^2 - h^2}{k^2 - h^2}\right)}\frac{d\omega}{\sin\omega}$$

Putting in this value for $d\nu$, we have

$$P^2 = \frac{(a^2 - h^2)\ (h^2 - \nu^2)\ (\mu'^2 - h^2)}{h^2(k^2 - h^2)\sin^2\omega} = \frac{b^2 b'^2 b''^2}{(b^2 - a^2)\ (b^2 - c^2)\sin^2\omega} = \frac{y^2}{\sin^2\omega}$$

(Art. 160); therefore $P = y \operatorname{cosec} \omega$.

In this investigation it is not necessary to assume the result of the last article. If we substitute for the right-hand side of the equation in the last article an undetermined function of ω, it is proved in like manner that $P = y\phi(\omega)$. We determine then the form of the function by remembering that in the neighbourhood of the umbilic the surface approaches to the form of a sphere. Now on a sphere the formula of rectification is $ds^2 = d\rho^2 + \sin^2\rho d\omega^2$. Hence $P = \sin\rho$. But in the sphere $y = \sin\rho \sin\omega$. The function therefore which multiplies y is $\operatorname{cosec}\omega$.

416. Consider now the triangle formed by joining any point P to the two umbilics O, O'. Then for the arc OP we have the function $P = y \operatorname{cosec}\omega$, and for the arc $O'P$, connecting P with the other umbilic, we have the function $P' = y \operatorname{cosec}\omega'$; and $P : P' :: \sin\omega' : \sin\omega$, an equation analogous to that which expresses that the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles, since P and P' in the rectification of arcs on the ellipsoid answer to $\sin\rho$, $\sin\rho'$ on the sphere.

417. Again, if P be any point on a line of curvature we know (Art. 400) $d\rho \pm d\rho' = 0$, where ρ and ρ' are the distances from the two umbilics. Now if θ be the angle which the

radius vector OP makes with the tangent, the perpendicular element $Pd\omega$ is evidently $d\rho$ tan θ. But the radius vector $O'P$ makes also the angle θ with the tangent. Hence we have

$$Pd\omega \pm P'd\omega' = 0, \text{ or } \frac{d\omega}{\sin\omega} \pm \frac{d\omega'}{\sin\omega'} = 0,$$

whence $\tan\frac{1}{2}\omega \tan\frac{1}{2}\omega'$ is constant when the sum of sides of the triangle is given; and $\tan\frac{1}{2}\omega$ is to $\tan\frac{1}{2}\omega'$ in a given ratio when the difference of sides of the triangle is given. Thus, then, the distance between two umbilics being taken as the base of a triangle, when either the product or the ratio of the tangents of the halves of the base angles is given, the locus of vertex is a line of curvature.*

From this theorem follow many corollaries: for instance, "if a geodesic through an umbilic O meet a line of curvature in points P, P' then (according to the species of the line of curvature) either the product or the ratio of $\tan\frac{1}{2}PO'O$, $\tan\frac{1}{2}P'O'O$, is constant". Again, "if the geodesics joining to the umbilics any point P on a line of curvature meet the curve again in P', P'', the locus of the intersection of the transverse geodesics $O'P'$, OP'' will be a line of curvature of the same species".

418. Roberts's expression for the element of an arc perpendicular to an umbilical geodesic has been extended as follows by Hart: Let OT, OT' be two consecutive geodesics touching the line of curvature formed by the intersection of the surface with a confocal B, $d\omega$ the angle at which they intersect; then the tangent at any point T of either geodesic touches B in a point P (Art. 405); and if TT' be taken conjugate to TP, the tangent plane at T' passes through TP (Art. 268), and the tangent line to the geodesic at T' touches the confocal B in the same point P. We want now to express

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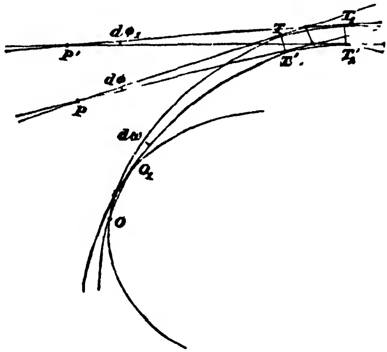
* This theorem, as well as those on which its proof depends (Art. 414, &c.), is due to M. Roberts, to whom this department of Geometry owes so much (*Liouville*, Vols. XIII. p. 1 and XV. p. 275).

in the form $Pd\omega$ the perpendicular distance from T' to TP. Let the tangents at consecutive points, one on each geodesic, intersect in P' and make with each other an angle $d\phi'$. Let normals to the surface on which the geodesics are drawn at the points T_1, T'_1, meet the tangents, PT, PT' at the points T_2, T'_2, then since the difference between $T_1T'_1$, $T_2T'_2$, is infinitely small of the third order, $PT_2d\phi$ and $P'T_1d\phi'$ are equal, to the same degree of approximation. But PT_2, $P'T_1$ are proportional to D and D', the diameters of the surface B drawn parallel to the two successive tangents to the geodesic. Hence $Dd\phi = D'd\phi'$. This quantity therefore



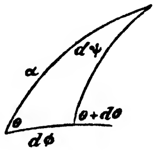
remains invariable as we proceed along the geodesic; but at the point O, $d\phi = d\omega$; if therefore D_0 be the diameter of B parallel to the tangent at O to the geodesic, $Dd\phi = D_0 d\omega$; and therefore the distance we want to express $PTd\phi = \frac{D_0}{D}\, td\omega$, where t ($= PT$) is the length of the tangent from T to the confocal B; or $\frac{D_0}{D}\, t$ is a mean between the segments of a chord of B drawn through T parallel to the tangent at O. When the geodesic passes through an umbilic, the surface B reduces to the plane of the umbilics, and $\frac{D_0}{D}\, t$ becomes the line drawn

through T to meet the plane of the umbilics parallel to the tangent at O, which is Roberts's expression

Hence, *If a geodesic polygon circumscribe a line of curvature, and if all the angles but one move on lines of curvature, this also will move on a line of curvature, and the perimeter of the polygon will be constant when the lines of curvature are of the same species* The proof is identical with that given for the corresponding property of plane conics (*Conics*, Art 401)* [*Cf* also, Ex 2, Art 421*b*]

419 If a geodesic joining any umbilic to that diametrically opposite, and making an angle ω with the plane of the umbilics, be continued so as to return to the first umbilic, it will not, as in the case of the sphere, then proceed on its former path, but after its return will make with the plane of the umbilics an angle different from ω In order to prove this we shall investigate an expression for θ, the angle made with the plane of the umbilics by the osculating plane at any point of that geodesic

It is convenient to prefix the following lemma In a spherical triangle let one side and the adjacent angle remain finite while the base diminishes indefinitely, it is required to find the limit of the ratio of the base to the difference of the base angles measured in the same direction The formula of spherical trigonometry



$\cos\frac{1}{2}(A+B)=\sin\frac{1}{2}C\,\dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}$ gives us in the limit $d\theta=\cos a\,d\psi$ But evidently $\sin a\,d\psi=\sin\theta\,d\phi$. Hence $\dfrac{d\theta}{\sin\theta}=\dfrac{d\phi}{\tan a}$

Now we know (Art 405) that the tangent line at any point of a geodesic passing through an umbilic, if produced goes to meet the plane of the umbilics in a point on the focal hyperbola, and the osculating plane of the geodesic at that point will be the plane joining the point to the corresponding

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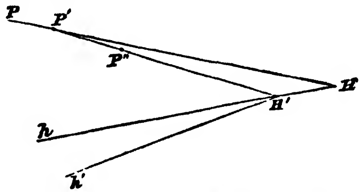
* See *Cambridge and Dublin Mathematical Journal*, Vol. IV p 192.

tangent of the focal hyperbola. We know also (Art. 184) that the cone circumscribing an ellipsoid, and whose vertex is any point on the focal hyperbola, is a right cone.

Let now PP' be an element of an umbilical geodesic produced to meet the focal hyperbola in H. Let $P'P''$ be the consecutive element meeting the focal hyperbola in H'; then if Hh, $H'h'$ be two consecutive tangents of the focal hyperbola, PHh, $P'H'h'$ will be two consecutive osculating planes. Imagine now a sphere round H', and consider the spherical



triangle formed by radii to the points h, h', P'. Then if $d\phi$ be the angle $hH'h'$, the angle of contact of the focal hyperbola; if θ be the angle between the osculating plane and $hH'h'$ the plane of the umbilics, while $hH'P'$ is a the semi-angle of the cone; the spherical triangle becomes that considered in our lemma, and we have $\dfrac{d\theta}{\sin\theta} = \dfrac{d\phi}{\tan a}$.

In order to integrate this equation we must express $d\phi$ in terms of a; and this we may regard as a problem in plane geometry, for a is half the angle included between the tangents from H to the principal section in the plane of the umbilics, while $d\phi$ is the angle of contact of the focal hyperbola at the same point. Now if a', b'; a'', b'' be the axes of an ellipse and hyperbola passing through H, confocal to an ellipse whose axes are a, b; and if $2a$ be the angle included between the tangents from H to the latter ellipse, we have (see *Conics*, Art. 194*d*) $\tan^2 a = \dfrac{a^2 - a''^2}{a'^2 - a^2}$. Differentiating, regarding a'' as constant (since we proceed to a consecutive point along the same confocal hyperbola), we have

$$d\alpha = -\tan\alpha\,\frac{a'da'}{a'^2-a''^2}.$$

But if p, p' be the central perpendiculars on the tangents at H to the ellipse and hyperbola, we have $a'da' = pd\sigma$ (Art. 410), where $d\sigma$ is the element of the arc of the focal hyperbola, and if ρ be the radius of curvature at the same point, $d\sigma = \rho d\phi$. But $\rho = \dfrac{a'^2-a''^2}{p'}$. Hence, $d\alpha = -\tan\alpha\dfrac{pd\phi}{p'}$ or $d\alpha = \tan\alpha\,\dfrac{a'b'd\phi}{a''b''}$.

But $a'^2 = a^2 + (a^2 - a''^2)\cot^2\alpha$, $b'^2 = b^2 + (a^2 - a''^2)\cot^2\alpha$.

$$\text{Hence } \frac{d\phi}{\tan\alpha} = \frac{a''b''d\alpha}{\sqrt{(a^2-a''^2+a^2\tan^2\alpha)}\sqrt{(a^2-a''^2+b^2\tan^2\alpha)}}.$$

In the case under consideration the axes of the touched ellipse are a, c; while the squares of the axes of the confocal hyperbola are a^2-b^2, b^2-c^2. Hence we have the equation

$$\frac{d\theta}{\sin\theta} = \frac{\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}\,d\alpha}{\sqrt{(b^2+a^2\tan^2\alpha)}\sqrt{(b^2+c^2\tan^2\alpha)}}.$$

Integrating this, and taking one limit of the integral at the umbilic where we have $\theta = \omega$, and $\alpha = \frac{1}{2}\pi$, we have

$$\log\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\omega} = \int_{\frac{1}{2}\pi}^{\alpha}\frac{\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}\,d\alpha}{\sqrt{(b^2+a^2\tan^2\alpha)}\sqrt{(b^2+c^2\tan^2\alpha)}}.$$

If, then, I be the value of this integral, we have $\tan\frac{1}{2}\theta = k\tan\frac{1}{2}\omega$, where $k = e^I$.

Now this integral obviously does not change sign between the limits $\pm\frac{1}{2}\pi$, that is to say, in passing from one umbilic to the other. If, then, ω' be the value of θ for the umbilic opposite to that from which we set out, at this limit I has a value different from zero, and k a value different from unity; and we have $\tan\frac{1}{2}\omega' = k\tan\frac{1}{2}\omega$; ω' is therefore always different from ω. And in like manner *the geodesic returns to the original umbilic, making an angle ω'' such that* $\tan\frac{1}{2}\omega'' = k^2\tan\frac{1}{2}\omega$, *and so it will pass and repass for ever, making a series of angles the tangents of whose halves are in continued proportion.**

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* The theorems of this article are Hart's, *Cambridge and Dublin Mathematical Journal*, Vol. IV. p. 82; but in the mode of proof I have followed William Roberts, *Liouville*, 1857, p. 213.

[It is proved (Ex. 1, Art. 421*b*) that *the lengths of these geodesics from umbilic to umbilic are constant.*]

420. If we consider edges belonging to the same tangent cone, whose vertex is any point H on the focal hyperbola, a (and therefore k) is constant; and the equation $\tan\frac{1}{2}\theta = k\tan\frac{1}{2}\omega$ gives $\frac{d\theta}{\sin\theta} = \frac{d\omega}{\sin\omega}$. Now since the osculating plane of the geodesic is normal to the surface, and therefore also normal to the tangent cone, it passes through the axis of that cone. If, then, we cut the cone by a plane perpendicular to the axis, the section is evidently a circle whose radius is $\frac{y}{\sin\theta}$, and the element of the arc is $\frac{yd\theta}{\sin\theta}$, or $\frac{yd\omega}{\sin\omega}$. Now this element, being the distance at their point of contact of two consecutive sides of the circumscribing cone, is what we have called (Art. 415) $Pd\omega$, and we have thus, from the investigation of the last article, an independent proof of the value found for P (Art. 415).

421. *Lines of level.* The inequalities of level of a country can be represented on a map by a series of curves marking the points which are on the same level. If a series of such curves be drawn, corresponding to equi-different heights, the places where the curves lie closest together evidently indicate the places where the level of the country changes most rapidly; the curve through the summit of a pass, or at the point of outflow of a lake, has this point for a node, &c., &c.* Generally, the curves of level of any surface are the sections of that surface by a series of horizontal planes, which we may suppose all parallel to the plane of xy. The equations of the horizontal projections of such a series are got by putting $z=c$ in the equation of the surface; and a differential equation common to all these projections is got

* See Reech, Sur les Surfaces Fermées, *Jour. de l'Ec. Polyt.* t. XXI. (1858), p. 169. Cayley on Contour and Slope Lines, *Phil. Mag.*, Vol. XVIII. 1859, p. 264.

by putting $dz = 0$ in the differential equation of the surface, when we have

$$U_1 dx + U_2 dy = 0.$$

We can make this a function of x and y only, by eliminating the z, which may enter into the differential coefficients, by the help of the equation of the surface.

Lines of greatest slope. The line of greatest slope through any point is the line which cuts all the lines of level perpendicularly; and the differential equation of its projection therefore is

$$U_1 dy - U_2 dx = 0.$$

The line of greatest slope is often defined as such that the tangent at every point of it makes the greatest angle with the horizon. Now it is evident that the line in any tangent plane which makes the greatest angle with the horizon is that which is perpendicular to the horizontal trace of that plane. And we get the same equation as before by expressing that the projection of the element of the curve (whose direction-cosines are proportional to dx, dy) is perpendicular to the trace whose equation is $U_1(x - x') + U_2(y - y') - U_3 z' = 0.$*

Ex. 1. To find the line of greatest slope on the quadric $Ax^2 + By^2 + Cz^2 = D$.

The differential equation is $Axdy = Bydx$, which integrated, gives $\left(\frac{x}{x'}\right)^B = \left(\frac{y}{y'}\right)^A$, where the constant has been determined by the condition that the line shall pass through the point $x = x'$, $y = y'$. The line of greatest slope is the intersection of the quadric by the cylinder whose equation has just been written, and will be a curve of double curvature, except when $x'y'$ lies in one of the principal planes when the equation just found reduces to $x = 0$ or $y = 0$.

Ex. 2. The coordinates of any point on the hyperboloid of one sheet may be written $\frac{x}{a} = \frac{1 + \lambda\mu}{\lambda + \mu}$, $\frac{y}{b} = \frac{\lambda - \mu}{\lambda + \mu}$, $\frac{z}{c} = \frac{1 - \lambda\mu}{\lambda + \mu}$; show that if $p = \frac{a^2 - 2b^2 - c^2}{a^2 + c^2}$ the lines of curvature are determined by the equations (cf. note, Art. 409)

* It is evident that the differential equation of the curve, which is always perpendicular to the intersection of the tangent plane [whose direction-cosines are as $L : M : N$] by a fixed plane whose direction-cosines are a, b, c, is

$$\begin{vmatrix} dx, & dy, & dz \\ L, & M, & N \\ a, & b, & c \end{vmatrix} = 0.$$

$$\frac{d\lambda}{\sqrt{(1 - 2p\lambda^2 + \lambda^4)}} \pm \frac{d\mu}{\sqrt{(1 - 2p\mu^2 + \mu^4)}} = 0$$

Ex 3 Express in the same system of coordinates the differential equation of geodesics on the surface

Staude's Constructions for Confocal Quadrics

[421*a* By the use of elliptic coordinates it is possible to prove Staude's extension to three dimensions of Graves's well-known theorem in conics We notice in the first place that an ellipsoid divides a confocal hyperboloid of one sheet into two visible portions (see illustration of confocals fig 4, ch v) separated by an invisible belt The line of curvature in which the two quadrics intersect consists of two congruent closed branches symmetrically placed with regard to the centre and plane of z and on opposite sides of this plane Staude* then states his theorem as follows

If a closed thread passed round the combination of an ellipsoid and a confocal one sheeted hyperboloid be stretched through a movable point P, so that it always touches each of the two visible portions of the hyperboloid, the point describes a confocal ellipsoid and the bisector of the angle between the directions of the thread at P is the normal to the ellipsoid at P

By "touching" is meant either touching at a single point, passing along a geodesic or touching (or passing along) the line of curvature in which the quadrics intersect The rectilineal portions of the string are two of the common tangent lines from P to the quadrics, namely either of the pairs whose common plane contains the normal to the confocal ellipsoid through P, this normal bisecting the angle between the corresponding tangent lines We recall also (Art 405) that the geodesic production of a common tangent line touches the line of curvature common to the two quadrics

It is convenient to represent the confocal system by the equation

$$\frac{x^2}{a-t} + \frac{y^2}{\beta - t} + \frac{z^2}{\gamma - t} = 1.$$

* Math Ann xx p 147 (1882)

Let λ, μ, ν be the values of t corresponding (at a point x, y, z) to the confocal ellipsoid, the hyperboloid of one sheet and the hyperboloid of two sheets (Art. 159), so that in general

$$\alpha > \nu > \beta > \mu > \gamma > \lambda > -\infty \quad . \quad . \quad . \qquad (1)$$

There are limiting values for λ, μ, ν; thus $\nu = \alpha$ represents the plane $x = 0$; $\nu = \beta$ the portions of the plane $y = 0$ to which the focal hyperbola is concave, $\mu = \beta$ the remaining portion of the same plane, and $\nu = \mu = \beta$ represents the focal hyperbola; $\mu = \gamma$ the portion of the plane $z = 0$ outside the focal ellipse, $\lambda = \gamma$ the inner portion of the same plane, while $\lambda = \mu = \gamma$ represents the focal ellipse (cf. Art. 158).

Let λ_0, μ_0 be the two fixed confocals. If P (λ, μ, ν) is outside the ellipsoid and outside the hyperboloid (i.e. in the region not containing the axis of z) then $\lambda_0 > \lambda$, and $\mu_0 > \mu$. (2)

The proof now depends on the principle that the elements of length (i) of a common tangent line from P (ii) of its geodesic production on either surface ($d\lambda = 0$ or $d\mu = 0$), and (iii) of the line of curvature ($d\lambda = 0$, $d\mu = 0$) or its geodesic productions, can be expressed collectively by the same formula, in which the variables are separated, viz.

$$ds = \pm \frac{d\lambda}{P} \pm \frac{d\mu}{Q} \pm \frac{d\nu}{R} \ldots \quad . \quad . \quad . \qquad (3)$$

$$\text{where } P \equiv \frac{2L}{(\lambda_0 - \lambda)\ (\mu_0 - \lambda)}$$

$$Q \equiv \frac{2M}{(\mu - \lambda_0)\ (\mu_0 - \mu)}$$

$$R \equiv \frac{2N}{(\nu - \lambda_0)\ (\nu - \mu_0)}$$

$$\text{and } L \equiv \sqrt{(\alpha - \lambda)\ (\beta - \lambda)\ (\gamma - \lambda)\ (\mu_0 - \lambda)\ (\lambda_0 - \lambda)}$$

and M and N have the values obtained by replacing λ by μ and ν, but leaving λ_0 and μ_0 intact.

Staude also proves that the signs to be given to the radicals P, Q, R in the expression (3), ds being supposed positive, are to be determined by the condition that each element is posi-

tive; * for example, if λ is decreasing $-\frac{d\lambda}{P}$ is the corresponding member in ds. All this amounts to saying that the sign of a radical (P, Q, or R) changes when we pass through a *limiting* value of the corresponding parameter (λ μ or ν), i.e. through

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* The proof being somewhat long, we refer to Staude's paper, where the whole subject is treated with great thoroughness. The method in outline is this:—By using the expressions for x^2, y^2, z^2 in terms of λ, μ, ν (cf. Art. 160) we find

$$ds^2 = \tfrac{1}{4}\Sigma \frac{(\mu-\lambda)(\nu-\lambda)}{(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)}\, d\lambda^2$$

where ds is any elementary length in space. By taking the tangent cones from any point, x', y', z', substituting in their equations dx, dy, dz for

$$x - x',\ y - y',\ z - z'$$

and expressing all in terms of λ, μ, ν we can prove that along any common tangent line of λ_0 and μ_0

$$\frac{d\lambda}{L(\mu-\nu)} = \frac{d\mu}{M(\nu-\lambda)} = \frac{d\nu}{N(\lambda-\mu)}, \quad . \quad . \quad . \quad . \quad (\text{i})$$

the signs (which correspond to the four tangent lines through a point) being undetermined. If we substitute in the equation for ds^2 it is easy to verify that

$$\pm\, ds = \frac{d\lambda}{P} - \frac{d\mu}{Q} + \frac{d\nu}{R} \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

the signs of P, Q, R being those of L, M, N in the preceding equation. We may assume that the positive direction of the string is that for which $\frac{d\nu}{R}$ is positive. In equation (i) $\frac{d\nu}{N}$ may be taken positive and therefore since $\nu > \mu > \lambda$, $\frac{d\lambda}{L}$ is positive, and $\frac{d\mu}{M}$ is negative. Hence in (ii) all the members on the right are essentially positive if $\frac{d\nu}{R}$ is positive. That is

$$ds = \pm\frac{d\lambda}{P} \pm \frac{d\mu}{Q} \pm \frac{d\nu}{R}$$

where each of the elements is given the sign which makes it positive.

The geodesics on λ_0 touching λ_0 μ_0 satisfy the equations (Art. 412)

$$\frac{d\mu}{M(\nu-\lambda_0)} = \frac{d\nu}{N(\lambda_0-\mu)} \quad . \quad . \quad . \quad . \quad . \quad (\text{iii})$$

This it should be noticed is the equation obtained by putting $\lambda = \lambda_0$ in (i). The element of arc is

$$ds = \pm\frac{d\mu}{Q} \pm \frac{d\nu}{R} \quad . \quad . \quad . \quad . \quad . \quad (\text{iv})$$

and we can show as before from equation (iii) that if $\pm\frac{d\nu}{R}$ is positive in (iv) then $\pm\frac{d\mu}{Q}$ is positive. The continuity in the transition from tangent line to

a point at which, or through a path along which, the variation of the parameter is zero.

The entire length of the string is then

$$s=\int_\lambda^\lambda \pm\frac{d\lambda}{P}+\int_\mu^\mu \pm\frac{d\mu}{Q}+\int_\nu^\nu \pm\frac{d\nu}{R}$$

where each element in each integral is always positive. The rest of the proof consists in showing that this integral depends only on λ if we proceed round the string.

Now λ has only one limiting value, $\lambda=\lambda_0$, since $\lambda=-\infty$ is obviously excluded. We assume that P, Q, R are themselves positive.

Thus the positive value of $\int_\lambda^\lambda \pm\frac{d\lambda}{P}$ is the positive value of

$$\int_\lambda^{\lambda_0}\frac{d\lambda}{P}-\int_{\lambda_0}^\lambda\frac{d\lambda}{P}$$

and this $=2\int_\lambda^{\lambda_0}\frac{d\lambda}{P}$ since $\lambda_0>\lambda$. Since λ cannot reach its other limit $-\infty$, we infer that the string having once left the ellipsoid cannot meet it again.

The value of $\int_\mu^\mu \pm\frac{d\mu}{Q}$ is the sum of the positive values of $\int\frac{d\mu}{Q}$ between P $(\mu=\mu)$ and the limiting values of μ, or between these limiting values. This sum if the string lies in the manner stated is the positive value of

$$\int_\mu^{\mu_0}dv-\int_{\mu_0}^\gamma dv+\int_\gamma^{\mu_0}dv-\int_{\mu_0}^\gamma dv+\int_\gamma^\mu dv$$

$=4\int_\gamma^{\mu_0}dv$ where $dv\equiv\frac{d\mu}{Q}$. Now $\mu=\gamma$ represents the plane

geodesic is evidently secured by supposing $d\mu$ and $d\nu$ to continue to have the same sign.

Similarly the element of arc of a line of curvature is

$$ds=\pm\frac{d\nu}{R}$$

where the left-hand side is essentially positive, and thus continuity is preserved as before.

Thus for all portions of the string

$$ds=\pm\frac{d\lambda}{P}\pm\frac{d\mu}{Q}\pm\frac{d\nu}{R}=0 \quad \text{(v)}$$

subject to the conditions mentioned.

$z=0$, $\mu=\mu_0$ is the hyperboloid, and it is easy to see that the path of the integral expresses the condition that the thread should "touch and again leave" the hyperboloid once on the upper side of $z=0$, and once again on the lower, "touching" being used in the general sense explained. That is the string touches both visible zones of the hyperboloid.

The limits of ν are α and β, and these give the coordinate planes x and y. Since the string passes completely round the hyperboloid it pierces each of these planes twice. Thus the positive value of $\int_\nu^\nu \pm\frac{d\nu}{R}$ is the positive value of

$$\int_\nu^\alpha dw-\int_\alpha^\beta dw+\int_\beta^\alpha dw-\int_\alpha^\beta dw+\int_\beta^\nu dw$$

and this $=4\int_\beta^\alpha dw$, where $dw\equiv\frac{d\nu}{R}$.

Thus *the entire length of the string is independent of μ and ν, and therefore the locus of P is a confocal ellipsoid $\lambda=$ constant.*

By examining the integrals we see that there are, in general, only two possible types of position for the string, corresponding to the two possible pairs of common tangent lines from P along which the string can lie. In one position the string first touches the hyperboloid, runs along the ellipsoid, meeting the opposite branch of the line of curvature, and returns to the rectilineal path from a geodesic on the ellipsoid. In the other position the string touches the ellipsoid first and continues thereon until it returns to the rectilineal path. On the way it touches both branches of the line of curvature.

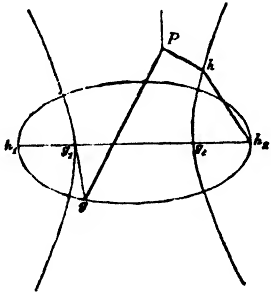
421*b*. Staude also considers the cases when one or both quadrics reduce to the focal ellipse or focal hyperbola. We shall examine a special instance of the case when the quadrics both reduce to focal conics.

Let g_1, g_2 be the foci of the focal ellipse G, h_1 and h_2 those of the focal hyperbola H. Then h_1 and h_2 are the vertices of G and g_1, g_2 those of H. Let h_1, g_1 lie on one side of the plane $x=0$. Putting $\lambda_0=\gamma$, $\mu_0=\beta$, the element of arc reduces to

$$ds = \pm\frac{d\lambda}{2\sqrt{a-\lambda}} \pm \frac{d\mu}{2\sqrt{a-\mu}} \pm \frac{d\nu}{2\sqrt{a-\nu}}$$

where each member on the right is to be positive. This formula gives the element of length of a common tangent line (e.g. Ph) to the two conics, or of its stretched continuation (e g. hh_2) from one conic to another, or of the line h_1h_2 which corresponds to the line of curvature.

Let P be a point in that quadrant of space which lies above the plane of the focal ellipse and in front of the plane of the focal hyperbola. We confine our attention to points and lines in this quadrant. Let r_1, r_2 be the "broken distances" from P over the focal ellipse to its foci; i.e. the paths of a stretched string; and let s_1, s_2 be the corresponding broken distances from P over the hyperbola to its foci. Then $r_1 = Pg + gg_1$, $s_2 = Ph + hh_2$.



We may assume that Pgg_1 pierces the plane $x=0$ (if it does not Phh_2 does, and the general result is the same). At $x=0$, $\nu=a$; at g, $\lambda=\mu=\gamma$; along gg_1, $d\lambda=0$; at g_1, $\lambda=\gamma$, $\mu=\nu=\beta$. Remembering that the differential involving a parameter changes sign after a limiting value of the latter, and that the parts involving each parameter are positive, we find

$$2r_1 = \int_\lambda^\gamma \frac{d\lambda}{\sqrt{a-\lambda}} - \int_\mu^\gamma \frac{d\mu}{\sqrt{a-\mu}} + \int_\gamma^\beta \frac{d\mu}{\sqrt{a-\mu}} + \int_\nu^a \frac{d\nu}{\sqrt{a-\nu}} - \int_a^\beta \frac{d\nu}{\sqrt{a-\nu}}$$

and the values of r_2, s_1, s_2 can be found similarly. Hence

$$\begin{aligned}
r_1 &= \sqrt{a-\lambda} - \sqrt{a-\mu} + \sqrt{a-\nu} + \sqrt{a-\gamma} \\
r_2 &= \sqrt{a-\lambda} - \sqrt{a-\mu} - \sqrt{a-\nu} + \sqrt{a-\gamma} \\
s_1 &= \sqrt{a-\lambda} + \sqrt{a-\mu} + \sqrt{a-\nu} - \sqrt{a-\beta} \\
s_2 &= \sqrt{a-\lambda} + \sqrt{a-\mu} - \sqrt{a-\nu} - \sqrt{a-\beta}
\end{aligned}$$

Hence $r_1 + s_2 = r_2 + s_1 = 2\sqrt{a-\lambda} + \sqrt{a-\gamma} - \sqrt{a-\beta}$ and is therefore *constant for a confocal ellipsoid.*

From this is derived a thread-construction for the confocal ellipsoids analogous to the construction for ellipses.*

Also $s_1 - r_1 = s_2 - r_2 = 2\sqrt{a-\mu} - \sqrt{a-\gamma} - \sqrt{a-\beta}$ and is therefore *constant for a confocal hyperboloid of one sheet.*

Finally since $r_1 - r_2 = s_1 - s_2 = 2\sqrt{a-\nu}$ these differences are *constant for a confocal two-sheeted hyperboloid.*

Using the notation of Art 160

$\sqrt{a-\lambda} = a'$, $\sqrt{a-\mu} = a''$, $\sqrt{a-\nu} = a'''$, $\sqrt{a-\beta} = h$, $\sqrt{a-\gamma} = k$
and thus we have four equations expressing the broken distances in terms of the semi-major axes of the confocals

$$r_1 = a' - a'' + a''' + k, \ \&c.$$

which are evidently the analogues of the two equations *in plano*

$$r_1 = a' - a'', \ r_2 = a' + a''$$

Ex. 1. Prove that *the length of a geodesic joining an umbilic to the opposite umbilic* (Art. 400) *is constant* and equals half the perimeter of the section by $y = 0$.

For such geodesics $\mu_0 = \beta$ (Art. 421a), since the confocal which they touch reduces to the focal hyperbola Hence (Note 421a)

$$2ds = \pm \frac{d\mu}{f(\mu)} \pm \frac{d\nu}{f(\nu)}$$

the signs being chosen so that each element on the right hand side is positive, and

$$\frac{1}{f(t)} = \frac{\sqrt{t-\lambda_0}}{\sqrt{(a-t)(t-\gamma)}}$$

Now at the umbilics, which lie on the focal hyperbola $\nu = \mu = \beta$ (Art. 421a), therefore ν always increases from the umbilic and μ diminishes, since $a > \nu > \beta > \mu > \gamma$. The lower limit of μ is γ ($z = 0$), and the upper limit of ν is a ($x = 0$) Hence, the radicals being positive

$$2s = -2\int_\beta^\gamma \frac{d\mu}{f(\mu)} + 2\int_\beta^a \frac{d\nu}{f(\nu)} = 2\int_\gamma^a \frac{dt}{f(t)}$$

The section by $y = 0$ is evidently a special case of such a geodesic.

Ex. 2. Prove the theorem at end of Art. 418, by integrating the expression for ds for a geodesic. See Note 421a (iv.).

* When the string $g_1 g\ Phh_1$ is stretched over the conics as in the figure, the locus of P is an ellipsoid

END OF VOL. I.

INDEX OF SUBJECTS.

(*For Index of Authors cited, see p.* 469).

INDEX OF AUTHORS CITED.

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